Contingency Tables & Goodness of Fit (8 pages; 12/5/17)

(A) Contingency Tables

(1) These are used to perform a hypothesis test about possible association between two factors, such as gender and voting.

We start with the table of observed frequencies, O_i (where, in this table, O_1 would be 45 and O_2 could be 37, with O_8 being 9; though O_2 could equally well have been 24).

O_i

	voting				
	Con	Lab	Lib	oth	
Male	45	37	12	6	100
Female	24	40	12	9	85
	69	77	24	15	185

This table is used to generate another table of expected frequencies, E_i. These are calculated on the basis that the null hypothesis (no association) is true. We reason that, as $\frac{69}{185}$ of the voters are Conservatives, we would expect there to be $\frac{69}{185} \times 100$ male Conservative voters, assuming that there is no association between gender and voting.

Another way of performing the calculation is to say that, of the total of 185 voters, $\frac{69}{185}$ of these are expected to be Conservatives, and of these $\frac{100}{185}$ are expected to be male, so that the expected number of male Conservative voters is $185 \times \frac{69}{185} \times \frac{100}{185}$

(this approach has the advantage that each E_i has the same form).

[Note that we could alternatively have taken the total of Conservative voters and multiplied by the proportion of male voters, to give $\frac{100}{185} \times 69$]

Applying this to each of the O_i, we thus obtain the table of expected frequencies:

E_i

	voting				
	Con	Lab	Lib	oth	
Male	37.297	41.622	12.973	8.108	100
Female	31.703	35.378	11.027	6.892	85
	69	77	24	15	185

(2) The next step is to measure the total deviation of the $O_i s$ from the $E_i s$. In order to avoid positive and negative deviations cancelling out, the $O_i - E_i$ are squared. We then divide by E_i, in order to obtain linear, rather than squared, quantities.

eg $\frac{(O_1 - E_1)^2}{E_1} = \frac{(45 - 37.297)^2}{37.297} = 1.5909$

and the total of these quantities is $\sum \frac{(O_i - E_i)^2}{E_i}$

and this has the value 5.9314 in this example, as shown in the table below.

i	O_i	E_i	$\frac{(O_i - E_i)^2}{E_i}$
1	45	32.297	1.5909
2	37	41.622	0.5133
3	12	12.923	0.0730
4	6	8.108	0.5481
5	24	31.703	1.8716
6	40	35.378	0.6038
7	12	11.027	0.0859
8	9	6.892	0.6948
			5.9314

Notes

(i) Another reason for including $(O_i - E_i)^2$ is that large differences are given a bigger weighting.

(ii) Dividing by E_i has the effect of standardising the squared differences: it takes account of the fact that if E_i is large then

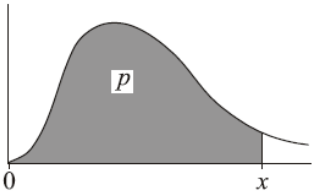
$O_i - E_i$ will also tend to be large.

(iii) When there are just two rows or two columns (or both), there is a case for having a separate column for $(O_i - E_i)^2$- as a check, as this value will be the same for the two cells in a particular column (row), where there are two rows (columns): if one of the O_i is larger than the corresponding E_i, then the other O_i in the same column must be smaller by the same amount, in order for the column total of the O_i to be the same as that of the E_i.

(3) The χ^2 distribution has been found to be a good model for the value of $X^2 = \sum \frac{(O_i - E_i)^2}{E_i}$, when there is no association between the factors. [Note the use of X^2, rather than χ^2, for the 'test statistic'.] From the χ^2 table (see below), we can find the 'critical value' of X^2; eg the value of X^2 that will only be exceeded in 5% of cases, when there is no association between the factors. [Note that, confusingly, the table refers to the χ^2 variable as X, rather than X^2.] The degrees of freedom are explained below.

TABLE 6 PERCENTAGE POINTS OF THE χ^2 DISTRIBUTION

The table gives the values of x satisfying $P(X \leqslant x) = p$, where X is a random variable having the χ^2 distribution with ν degrees of freedom.



p	0.005	0.01	0.025	0.05	0.1	0.9	0.95	0.975	0.99	0.995	p
ν											ν
1	0.00004	0.0002	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879	1
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597	2
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838	3
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860	4
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750	5
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548	6
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278	7
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955	8
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589	9
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188	10

(4) Degrees of freedom

Various statistical distributions depend on a parameter known as the degrees of freedom.

Clearly when the observed and expected values are being compared in a Contingency table, they are not entirely independent, as the row and column totals of the observed values have been used to determine the expected values. The degrees of freedom measure the extent of independence. Had the expected

values been determined without reference to the observed values, then, in the case of an $m \times n$ table, all mn terms of X^2 would be independent.

To obtain the degrees of freedom (d.f. - denoted by ν [nu]), we deduct 1 from mn for each constraint.

The deductions are as follows:

grand total fixed: -1

row totals fixed $-(m-1)$ [last one covered by grand total]

column totals fixed $-(n-1)$ [last one covered by grand total]

This gives $\nu = mn - 1 - (m-1) - (n-1)$

$= mn - m - n + 1 = (m-1)(n-1)$

(5) Hypothesis Test

For the above example:

H_0: there is no association between gender and voting habits

H_1: there is some association

significance level: eg 5%

Reject H_0 if $X^2 = \sum \frac{(O_i - E_i)^2}{E_i} > \chi^2{}_\nu$

In the example above, $\nu = (2-1)(4-1) = 3$,

so that the critical value, $\chi^2{}_\nu$ (at the 5% level) is 7.815

As $X^2 = 5.9314 < 7.815$, we accept H_0, and conclude that there isn't sufficient evidence of an association between gender and voting habits.

Note: The χ^2 test is one-tailed; ie the right-hand critical values are the ones that are used when performing hypothesis tests. However, a very low value of X^2 could be regarded as suspicious (ie the data might have been rigged). In theory, this could be tested for using the left-hand critical values from the table.

(6) Special Situations

(i) Small expected values

The test becomes unreliable when any of the E_i are less than 5.

To get round this, factors (of similar type) need to be grouped together. Thus, in the above example, had the "Female/Others" cell had an expected value less than 5, then we might group the Liberal and Others factors together. There would be no justification, however, for combining Conservatives and Others (as the Conservatives are not a minor party).

(ii) Yates' Correction for 2×2 table

The χ^2 model is less accurate for 2×2 tables, and the following adjusted value for X^2 has been found to be more appropriate:

$$\sum \frac{(|O_i - E_i| - 0.5)^2}{E_i}$$

Note that, in the case of a 2×2 table, all the values of

$(|O_i - E_i| - 0.5)^2$ will be the same.

(7) Notes

(i) Once H_1 has been accepted, values of $\frac{(O_i-E_i)^2}{E_i}$ for particular cells may suggest a theory as to the nature of an association between factors.

(ii) Association does not imply "cause & effect". For example, level of education and future wealth: although there is highly likely to be cause and effect here, there could also be indirect factors connected with genetics or upbringing, which have an impact on both level of education and future wealth.

(iii) It can be shown that the χ^2 distribution with ν degrees of freedom is in fact the distribution of ${Z_1}^2 + {Z_2}^2 + \cdots + {Z_\nu}^2$, where the Z_i are independent standardised Normal variables.

(B) Goodness of Fit Test

χ^2 tables are used in a similar way here. Instead of calculating expected frequencies from the row and column totals of the Contingency table, they are obtained from an supposed model distribution.

Example: To test whether a die is biased, we could roll it 600 times, and observe the numbers of 1s, 2s etc. There would be 6 cells or 'classes', and the expected frequency in each case would be 100.

Then $X^2 = \sum \frac{(O_i-E_i)^2}{E_i}$, as before.

In general, the E_i are obtained by multiplying the total observed frequency by the probability for class i, according to the model distribution.

The number of degrees of freedom is the number of free variables, which is the number of classes less the number of restrictions.

Here there is one restriction: the total of the observed frequencies has to equal 600.

Thus, $\nu = 6 - 1 = 5$

If, in addition, the observed values are used to estimate parameters of the model distribution, then the degrees of freedom will be given by:

$\nu = no.of\ classes - 1 - no.of\ parameters\ estimated$

In general, the null and alternative hypotheses would be:

H_0: the data are drawn from the model population

H_1: this is not the case

Notes

(i) For continuous data, observations would need to be grouped into suitable classes. Discrete data may also be grouped.

(ii) One use of the Goodness of Fit test is in establishing whether data come from a Normal distribution (for example, when deciding whether a t-test is appropriate for a small sample). The mean and variance of the supposed Normal distribution are estimated from the data.

However, if a t-test is subsequently carried out, it will be necessary to obtain another sample, as the original sample is not entirely random: it is a sample that passes the χ^2 test for Normality.

(iii) Class widths need not be the same. As a rule of thumb, it is best to choose them in such a way that the expected frequencies are in the range $8 - 12$. Thus the class width would be narrower where the model distribution has the greatest density.