Conics - Exercises (Solutions) (5 pages; 16/1/17)

(1) Show that the equation of the tangent to the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(acosht, bsinht)$ is

$yasinht = xbcosht - ab$

Solution

Using the parametric equations $x = acosht$ & $y = bsinht$,

$\frac{dx}{dt} = asinht$ & $\frac{dy}{dt} = bcosht$,

so that $\frac{dy}{dx} = \frac{bcosht}{asinht}$

and the equation of the tangent at $(acosht, bsinht)$ is

$\frac{y-bsinht}{x-acosht} = \frac{bcosht}{asinht}$

and hence $yasinht - absinh^2t = xbcosht - abcosh^2t$,

so that $yasinht = xbcosht - ab$

(2) Given that the tangent in (1) meets the asymptotes of the hyperbola at the points P & Q, show that the mid-point of P & Q is $(acosht, bsinht)$.

Solution

The asymptotes of the hyperbola are $y = \pm\frac{b}{a}x$

From (1), the tangent to the hyperbola at $(acosht, bsinht)$ meets the asymptote $y = \frac{b}{a}x$ at P (say), where $bxsinht = xbcosht - ab$ and the asymptote $y = -\frac{b}{a}x$ at Q where

$-bxsinht = xbcosht - ab$

so that P is the point $\left(\frac{a}{cosht-sinht}, \frac{b}{cosht-sinht}\right)$

and Q is the point $\left(\frac{a}{cosht+sinht}, \frac{-b}{cosht+sinht}\right)$

The mid-point of P & Q is therefore

$\left(\frac{1}{2}\left[\frac{a}{cosht-sinht}+\frac{a}{cosht+sinht}\right], \frac{1}{2}\left[\frac{b}{cosht-sinht}+\frac{-b}{cosht+sinht}\right]\right)$

$=\left(\frac{acosht}{cosh^2t-sinh^2t}, \frac{bsinht}{cosh^2t-sinh^2t}\right)=(acosht, bsinht)$, as required.

(3) In the case where $b=a$, find the area of the triangle OPQ (where P & Q are as in (2), and O is the Origin).

Solution

The two asymptotes are at right angles to each other, so that the required area, $A=\frac{1}{2}OP.OQ$

Then $4A^2=\left(\left(\frac{a}{cosht-sinht}\right)^{\mathbf{2}}+\left(\frac{a}{cosht-sinht}\right)^{\mathbf{2}}\right)$

$\times\left(\left(\frac{a}{cosht+sinht}\right)^{\mathbf{2}}+\left(\frac{-a}{cosht+sinht}\right)^{\mathbf{2}}\right)$

$=\left(\frac{2a^2}{(cosht-sinht)^2}\right)\left(\frac{2a^2}{(cosht+sinht)^2}\right)$

$=\frac{4a^4}{(cosh^2t-sinh^2t)^2}=4a^4$

and therefore $A=a^2$

(4) Given the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and circle $x^2+y^2=a^2$, let l_1 be the tangent to the ellipse at the point $(acos\theta, bsin\theta)$ and l_2 be the tangent to the circle at the point $(acos\theta, asin\theta)$. Find the locus of the point of intersection of l_1 & l_2, as θ varies.

Solution

The equation of l_1 is $\frac{y-bsin\theta}{x-acos\theta} = \frac{dy/d\theta}{dx/d\theta} = \frac{bcos\theta}{-asin\theta}$ (1)

The equation of l_2 is $\frac{y-asin\theta}{x-acos\theta} = \frac{dy/d\theta}{dx/d\theta} = \frac{acos\theta}{-asin\theta}$ (2)

At the intersection of l_1 & l_2,

$x - acos\theta = \frac{-asin\theta}{bcos\theta}(y - bsin\theta)$ from (1)

and $x - acos\theta = \frac{-sin\theta}{cos\theta}(y - asin\theta)$ from (2),

so that $\left(\frac{a}{b}\right)(y - bsin\theta) = y - asin\theta$

$\Rightarrow ay - absin\theta = by - absin\theta$

$\Rightarrow y = 0,$ as $a \neq b$ (otherwise the ellipse would be a circle)

Then, from (2), $x - acos\theta = \frac{asin^2\theta}{cos\theta}$,

so that $xcos\theta = acos^2\theta + asin^2\theta = a$, and thus $x = \frac{a}{cos\theta}$

As $-1 < cos\theta < 1$, x can take values in the range
$(-\infty, -a]$ & $[a, \infty)$

Thus the required locus is the set of points on the x-axis in the above range.

(5) The chord PQ, where P and Q are points on the rectangular hyperbola $xy = c^2$, has gradient 1. Show that the locus of the point of intersection of the tangents from P and Q is the line

$y = -x$. [Edx FP3 textbook, Ex. 2G, Q9]

Solution

Let P & Q be the points $\left(ct_1,\frac{c}{t_1}\right)$ & $\left(ct_2,\frac{c}{t_2}\right)$, respectively.

As the gradient of PQ is 1, $\frac{\frac{c}{t_2}-\frac{c}{t_1}}{ct_2-ct_1}=1$, so that

$\frac{1}{t_2}-\frac{1}{t_1}=t_2-t_1$

$\Rightarrow \frac{t_1-t_2}{t_1t_2}=t_2-t_1$

$\Rightarrow t_1t_2=-1$, as $t_1\neq t_2$ (P & Q being distinct points)

The equation of the tangent from P is:

$\frac{y-\frac{c}{t_1}}{x-ct_1}=\frac{dy/dt}{dx/dt}|t=t_1$, where $x=ct$ & $y=\frac{c}{t}$

so that $dy/dt=-\frac{c}{t^2}$ & $dx/dt=c$

and the equation of the tangent from P is

$\frac{y-\frac{c}{t_1}}{x-ct_1}=\frac{\left(-\frac{c}{t_1{}^2}\right)}{c}\Rightarrow t_1{}^2y-t_1c=-(x-ct_1)$

$\Rightarrow t_1{}^2y=-x+2ct_1 \quad (1)$

Similarly, the equation of the tangent from Q is $t_2{}^2y=-x+2ct_2$

and these lines intersect where

$t_1{}^2y-2ct_1=t_2{}^2y-2ct_2$,

so that $y(t_1{}^2-t_2{}^2)=2c(t_1-t_2)$

and $y=\frac{2c}{t_1+t_2}$ (as $t_1\neq t_2$)

Then, from (1), $x=2ct_1-\frac{2ct_1{}^2}{t_1+t_2}$

$= \frac{2ct_1{}^2+2ct_1t_2-2ct_1{}^2}{t_1+t_2}$

$= \frac{2ct_1t_2}{t_1+t_2}$

and so $\frac{y}{x} = \frac{1}{t_1t_2} = -1$ (found earlier),

and thus the points of intersection satisfy $y = -x$, as required.

(6) Using the parametric equations of a parabola ($x = at^2$, $y = 2at$), show that the midpoints of chords of a parabola that have the same direction lie on a straight line parallel to the x-axis.

[A chord of a parabola joins two points on the parabola.]

Solution

Let points P and Q on the parabola have parameters t_1 & t_2.

The chord PQ has gradient $\frac{2at_2-2at_1}{at_2{}^2-at_1{}^2} = \frac{2(t_2-t_1)}{t_2{}^2-t_1{}^2} = \frac{2}{t_1+t_2}$, and we are told that this is constant.

The y-coordinate of the midpoint of PQ is $\frac{1}{2}(2at_1 + 2at_2) = a(t_1 + t_2)$, which is constant, as $\frac{2}{t_1+t_2}$ and therefore $t_1 + t_2$ are constant, giving the required result.