Complex Numbers - Part 2 (12 pages; 4/6/23)

Correction : The imaginary axis of the Argand diagram is labelled $i, 2i, 3i, \dots$ in these notes. I would now recommend $1, 2, 3, \dots$ instead.

(14) Operations with conjugates

(i) Clearly $(z^*)^* = z$

(ii) Let $u = a + bi \quad and \quad v = c + di$

Then $(u + v)^* = (a + bi + c + di)^* = (a + c + [b + d]i)^*$

$= a + c - [b + d]i = (a - bi) + (c - di) = u^* + v^*$

ie $(u + v)^* = u^* + v^*$

and similarly, $(u - v)^* = u^* - v^*$

(iii) Also, $(a + bi)(c + di) = ac - bd + (bc + ad)i$

and $(a - bi)(c - di) = ac - bd - (bc + ad)i$

so that $(uv)^* = u^* v^*$ (A)

When $u = v = z$, $(z^2)^* = (z^*)^2$

and this can be extended to $(z^n)^* = (z^*)^n$

(iv) $(A) \Rightarrow \frac{(uv)^*}{v^*} = u^*$

If we let $u = \frac{p}{q} \quad and \quad v = q$, then $\left(\frac{p}{q}\right)^* = \frac{p^*}{q^*}$

As a special case, $\left(\frac{1}{z}\right)^* = \frac{1}{z^*}$

(15) Polynomial Equations

Let $p(z) = az^3 + bz^2 + cz + d$ (where a, b, c & d are real)

Then $(p(z))^* = (az^3)^* + (bz^2)^* + (cz)^* + d^*$

$= a(z^*)^3 + b(z^*)^2 + cz^* + d$

$= p(z^*)$

So, if $p(z) = 0$, $p(z^*) = (p(z))^* = 0^* = 0$

Hence, if α is a root of $p(z) = 0$,where $p(z)$ is a polynomial with real coefficients, then α^* is also a root.

Example: If $2 + i$ is a root of the equation

$x^3 - 7x^2 + 17x - 15 = 0$, find the remaining roots

First of all, the conguate of $2 + i$; ie $2 - i$ is a root

So $x^3 - 7x^2 + 17x - 15 = (x - [2 + i])(x - [2 - i])(x - \alpha)$

$= ([x - 2] - i])([x - 2] + i])(x - \alpha)$

$= [(x - 2)^2 - i^2](x - \alpha)$

$= (x^2 - 4x + 5)(x - \alpha)$

Then equating the constant terms, $\alpha = 3$

(16) Because (non-real) complex roots come in conjugate pairs, the possibilities for the roots of cubic and quartic equations are as follows:

(a) cubic equation

3 real roots

1 real root & 2 complex roots (conjugate pair)

(b) quartic equation

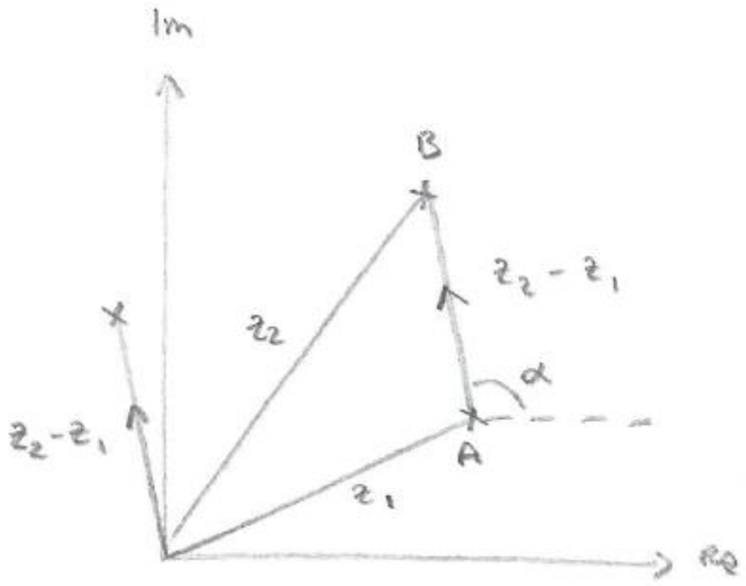
4 real roots

2 real roots & 2 complex roots (conjugate pair)

4 complex roots (in conjugate pairs)

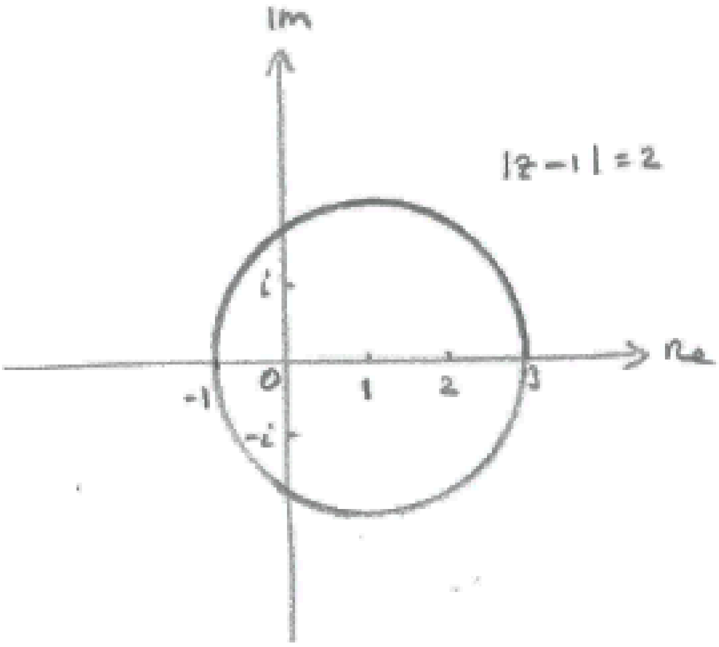
(17) $\boldsymbol{z_2 - z_1}$

By treating complex numbers in the Argand diagram as vectors, we see that:



$|z_2 - z_1| = |AB|$ and $\arg(z_2 - z_1) = \alpha$ (from the diagram below)

Problems involving moduli and arguments of complex numbers can often be converted to problems in geometry.



(18) Loci involving moduli

A **locus** is a collection of points satisfying a particular equation. The points will generally form a continuous curve.

Example: $|z-1|=2$

Approach 1: z must be a distance of 2 from $1+0i,$ and so the locus is that of a circle (see diagram below)

Approach 2

Let $z = x + yi$

Then $|z - 1| = 2 \;\Rightarrow\; |x - 1 + yi| = 2$,

so that $\sqrt{(x-1)^2 + y^2} = 2$,

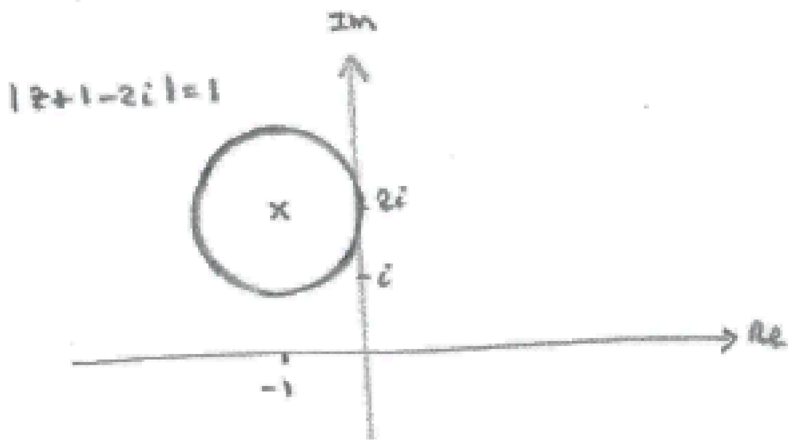
and hence $(x-1)^2 + y^2 = 4$

Exercise: Represent $|z + 1 - 2i| = 1$ on the Argand diagram, and demonstrate this algebraically.

Solution

$|z + 1 - 2i| = 1 \Rightarrow |z - (-1 + 2i)| = 1$

ie a circle of radius 1, centre $-1 + 2i$



Algebraically, $|z - (-1 + 2i)|^2 = 1$

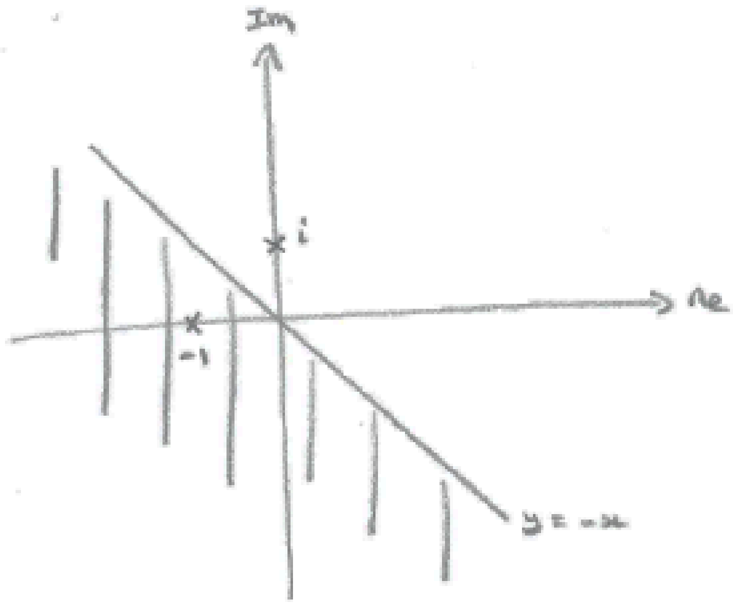
$\Rightarrow |x + 1 + (y - 2)i|^2 = 1$, if $z = x + yi$

$\Rightarrow (x + 1)^2 + (y - 2)^2 = 1$

Example: Represent the inequality $|z-i|>|z+1|$ on the Argand diagram

The requirement is for z to be further from i than it is from -1

(writing $|z+1|$ as $|z-(-1)|$, as usual). This gives the shaded area in the diagram below. The border of this area is the perpendicular bisector of the line joining the points i and -1.



Algebraically: Let $z=x+yi$

$|x+(y-1)i|^2>|(x+1)+yi|^2$

$\Rightarrow x^2+(y-1)^2>(x+1)^2+y^2$

$\Rightarrow -2y>2x$

$\Rightarrow y<-x$

Exercise: Represent $|z-i|=2|z+1|$ on an Argand diagram

This situation is much harder to visualise. Applying an algebraic approach:

Let $z=x+yi$

Then $|x+(y-1)i|^2=4|(x+1)+yi|^2$

$\Rightarrow x^2 + (y-1)^2 = 4\{(x+1)^2 + y^2\}$

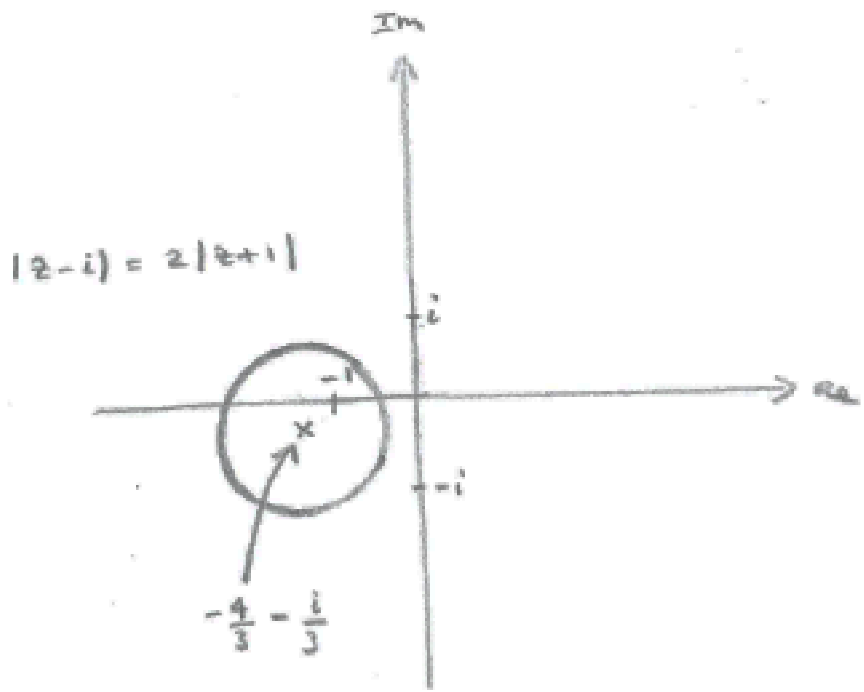
$\Rightarrow 3x^2 + 8x + 3y^2 + 2y + 3 = 0$

$\Rightarrow x^2 + \frac{8x}{3} + y^2 + \frac{2y}{3} + 1 = 0$

$\Rightarrow (x + \frac{4}{3})^2 + (y + \frac{1}{3})^2 - \frac{16}{9} - \frac{1}{9} + 1 = 0$

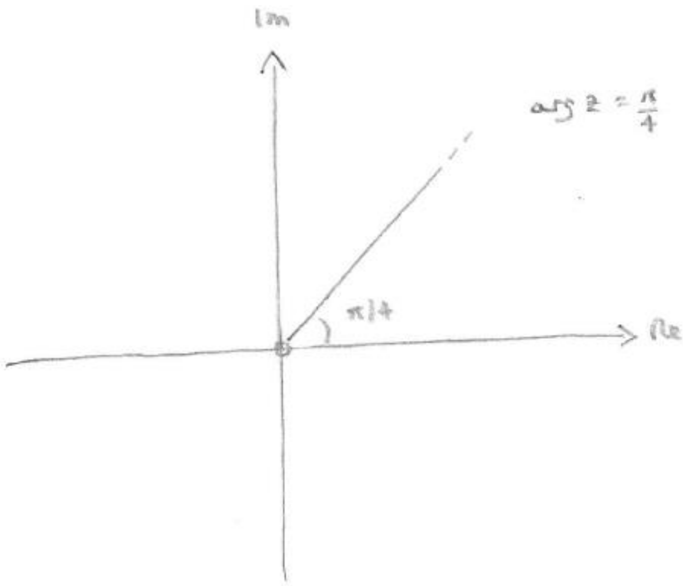
$\Rightarrow (x + \frac{4}{3})^2 + (y + \frac{1}{3})^2 = \frac{8}{9}$

ie a circle centre $\left(-\frac{4}{3}, -\frac{1}{3}\right)$, radius $\frac{2\sqrt{2}}{3}$



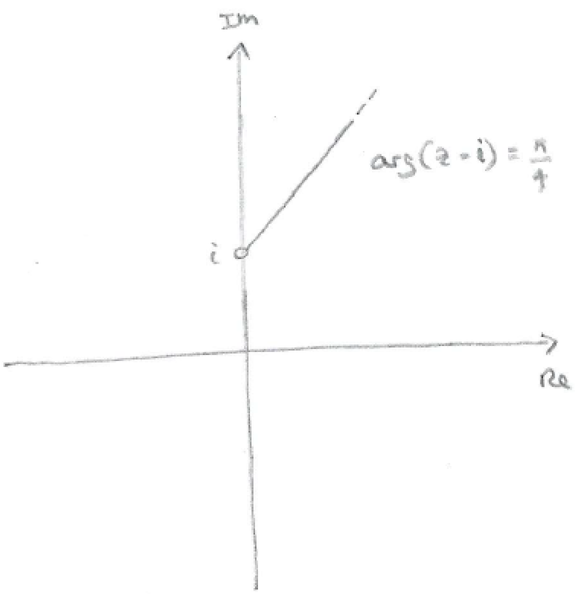
(19) Loci involving arguments

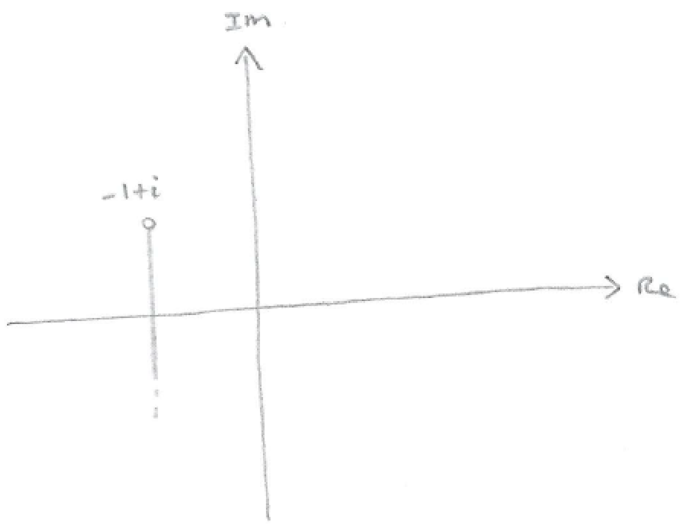
Example: $\arg z = \frac{\pi}{4}$



Note: The Origin is excluded, as $arg(0)$ is undefined

Example: $\arg(z - i) = \dfrac{\pi}{4}$

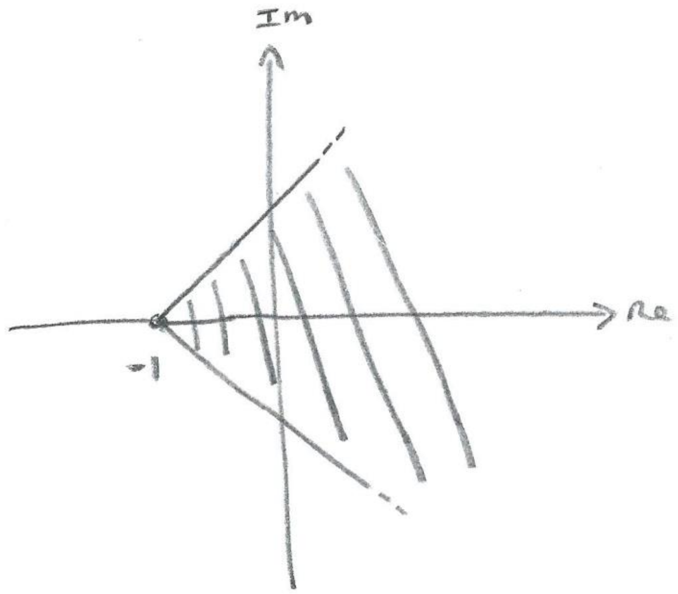




Exercise: Draw the locus of $\arg(z+1-i)=-\frac{\pi}{2}$

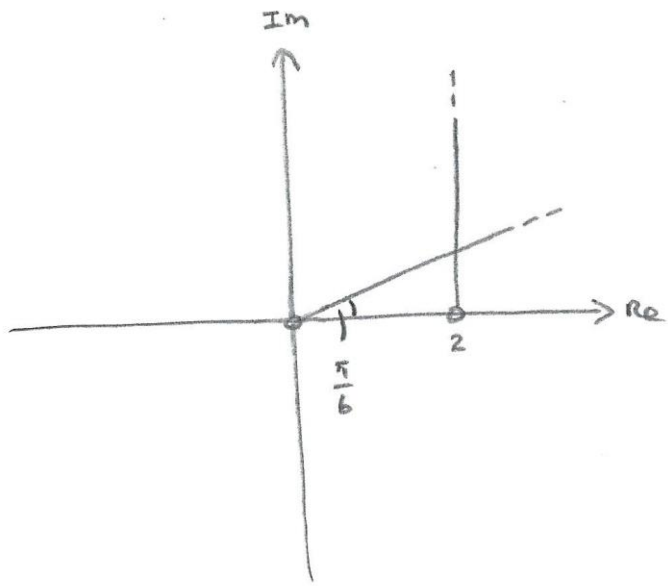
Rewriting as $\arg(z-[-1+i])=-\frac{\pi}{2}$:

Exercise: Show in an Argand diagram the set of points satisfying the inequality $-\frac{\pi}{4}\le \arg(z+1)\le \frac{\pi}{4}$



Example: Solve the simultaneous equations:

$\arg(z-2)=\frac{\pi}{2}$ and $\arg z=\frac{\pi}{6}$

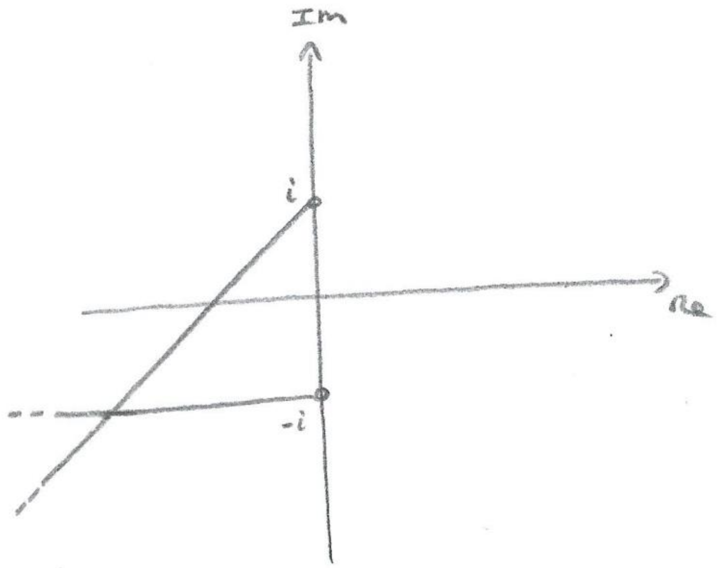


$\Rightarrow z = 2 + 2\tan\left(\frac{\pi}{6}\right)i$

$= 2 + \frac{2}{\sqrt{3}}i$ or $2 + \frac{2\sqrt{3}}{3}i$

Exercise: Solve the simultaneous equations:

$\arg(z+i)=\pi$ and $\arg(z-i)=\frac{-3\pi}{4}$



$\Rightarrow z = -2 - i$