# Complex Numbers - Part 2 (19 pages; 9/12/15)

## (14) Operations with conjugates

(i) Clearly $(z^*)^* = z$

(ii) Let $u = a + bi \quad and \quad v = c + di$

Then $(u + v)^* = (a + bi + c + di)^* = (a + c + [b + d]i)^*$

$= a + c - [b + d]i = (a - bi) + (c - di) = u^* + v^*$

ie $(u + v)^* = u^* + v^*$

and similarly, $(u - v)^* = u^* - v^*$

(iii) Also, $(a + bi)(c + di) = ac - bd + (bc + ad)i$

and $(a - bi)(c - di) = ac - bd - (bc + ad)i$

so that $(uv)^* = u^*v^*$ (A)

When $u = v = z$, $(z^2)^* = (z^*)^2$

and this can be extended to $(z^n)^* = (z^*)^n$

(iv) $(A) \Rightarrow \frac{(uv)^*}{v^*} = u^*$

If we let $u = \frac{p}{q} \quad and \quad v = q$ , then $\left(\frac{p}{q}\right)^* = \frac{p^*}{q^*}$

As a special case, $\left(\frac{1}{z}\right)^* = \frac{1}{z^*}$

## (15) Polynomial Equations

Let $p(z) = az^3 + bz^2 + cz + d$ (where a, b, c & d are real)

Then $(p(z))^* = (az^3)^* + (bz^2)^* + (cz)^* + d^*$

$= a(z^*)^3 + b(z^*)^2 + cz^* + d$

$= p(z^*)$

So, if $p(z) = 0$, $p(z^*) = (p(z))^* = 0^* = 0$

Hence, if $\alpha$ is a root of $p(z) = 0$ ,where $p(z)$ is a polynomial with real coefficients, then $\alpha^*$ is also a root.

**Example**: If $2 + i$ is a root of the equation

$x^3 - 7x^2 + 17x - 15 = 0$, find the remaining roots

First of all, the conguate of $2 + i$ ; ie $2 - i$ is a root

So $x^3 - 7x^2 + 17x - 15 = (x - [2 + i])(x - [2 - i])(x - \alpha)$

$= ([x - 2] - i])([x - 2] + i])(x - \alpha)$

$= [(x - 2)^2 - i^2](x - \alpha)$

$= (x^2 - 4x + 5)(x - \alpha)$

Then equating the constant terms, $\alpha = 3$

(16) Because (non-real) complex roots come in conjugate pairs, the possibilities for the roots of cubic and quartic equations are as follows:

(a) cubic equation

3 real roots

1 real root & 2 complex roots (conjugate pair)

(b) quartic equation

4 real roots

2 real roots & 2 complex roots (conjugate pair)

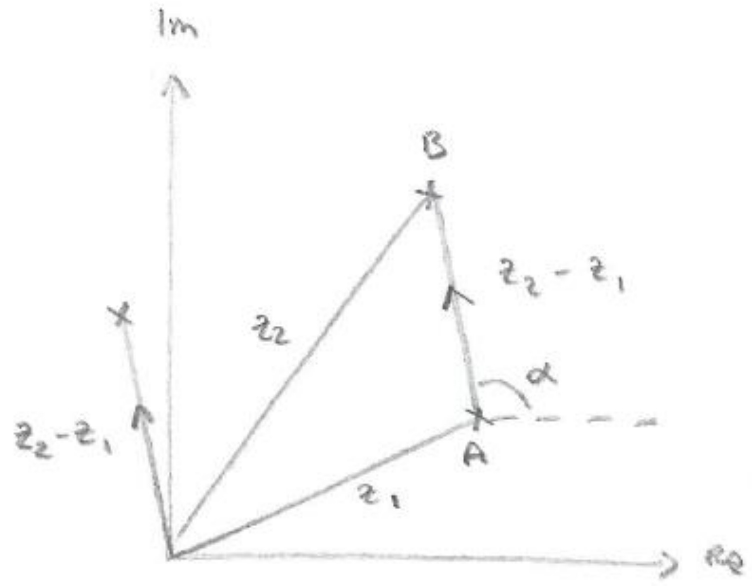
4 complex roots (in conjugate pairs)

## (17) $z_2 - z_1$

By treating complex numbers in the Argand diagram as vectors, we see that:

$|z_2 - z_1| = |AB|$ and $\arg(z_2 - z_1) = \alpha$ (from the diagram below)

Problems involving moduli and arguments of complex numbers can often be converted to problems in geometry.

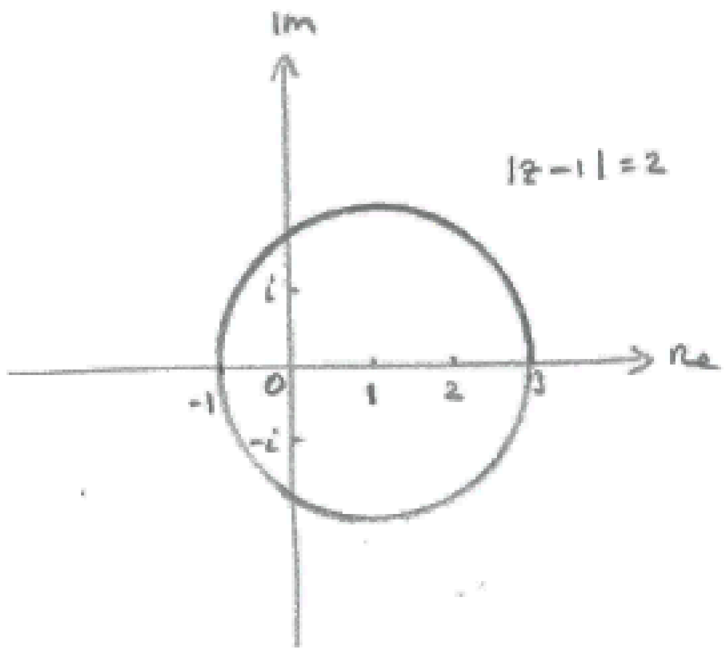



## (18) Loci involving moduli

A **locus** is a collection of points satisfying a particular equation. The points will generally form a continuous curve.

**Example**: $|z - 1| = 2$

**Approach 1**: z must be a distance of 2 from $1 + 0i$, and so the locus is that of a circle (see diagram below)




**Approach 2**

Let $z = x + yi$

Then $|z - 1| = 2 \;\Rightarrow\; |x - 1 + yi| = 2$,

so that $\sqrt{(x-1)^2 + y^2} = 2$ ,

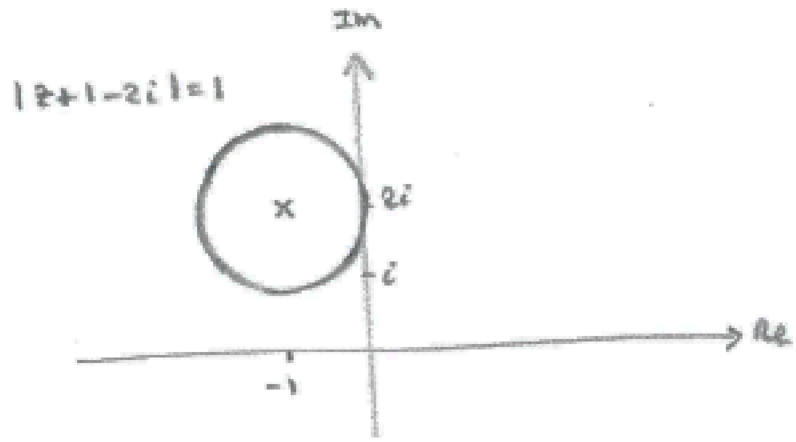
and hence $(x-1)^2 + y^2 = 4$

**Exercise**: Represent $|z + 1 - 2i| = 1$ on the Argand diagram, and demonstrate this algebraically.

**Solution**

$|z + 1 - 2i| = 1 \Rightarrow |z - (-1 + 2i)| = 1$

ie a circle of radius 1, centre $-1 + 2i$




Algebraically, $|z-(-1+2i)|^2 = 1$

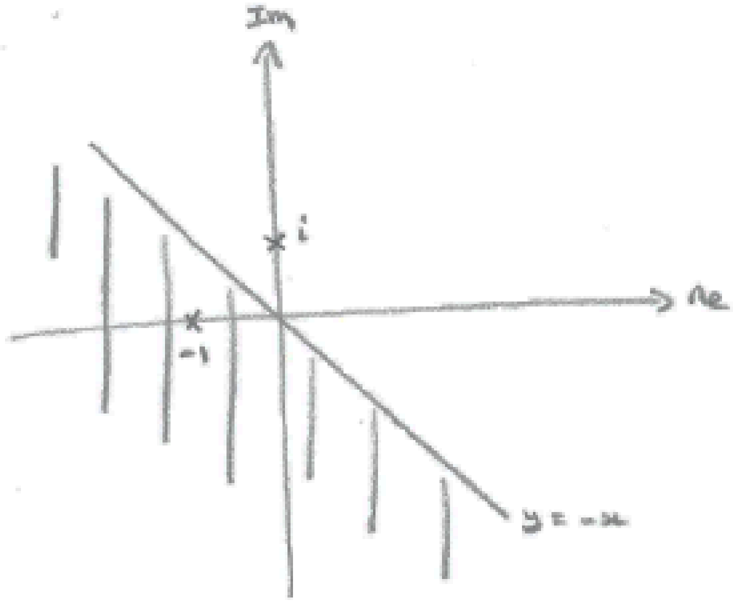
$\Rightarrow |x+1+(y-2)i|^2 = 1$, if $z = x + yi$

$\Rightarrow (x+1)^2 + (y-2)^2 = 1$

**Example:** Represent the inequality $|z-i| > |z+1|$ on the Argand diagram

The requirement is for $z$ to be further from $i$ than it is from $-1$

(writing $|z+1|$ as $|z-(-1)|$, as usual). This gives the shaded area in the diagram below. The border of this area is the perpendicular bisector of the line joining the points $i$ and $-1$.




Algebraically: Let $z = x + yi$

$|x + (y-1)i|^2 > |(x+1) + yi|^2$

$\Rightarrow x^2 + (y-1)^2 > (x+1)^2 + y^2$

$\Rightarrow -2y > 2x$

$\Rightarrow y < -x$

**Exercise:** Represent $|z - i| = 2|z + 1|$ on an Argand diagram

This situation is much harder to visualise. Applying an algebraic approach:

Let $z = x + yi$

Then $|x + (y-1)i|^2 = 4|(x+1) + yi|^2$

$\Rightarrow x^2 + (y-1)^2 = 4\{(x+1)^2 + y^2\}$

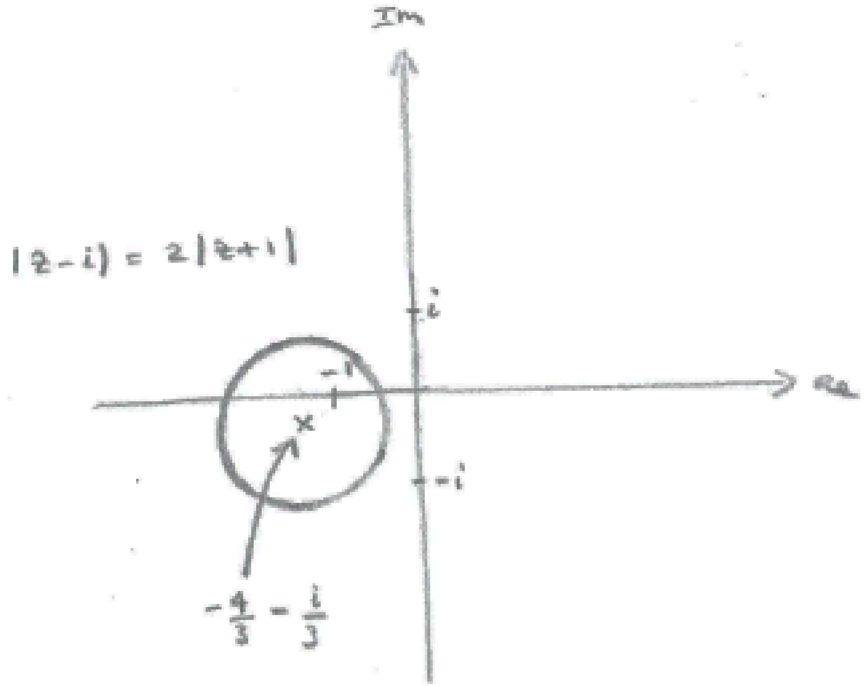
$\Rightarrow 3x^2 + 8x + 3y^2 + 2y + 3 = 0$

$\Rightarrow x^2 + \frac{8x}{3} + y^2 + \frac{2y}{3} + 1 = 0$

$\Rightarrow (x + \frac{4}{3})^2 + (y + \frac{1}{3})^2 - \frac{16}{9} - \frac{1}{9} + 1 = 0$

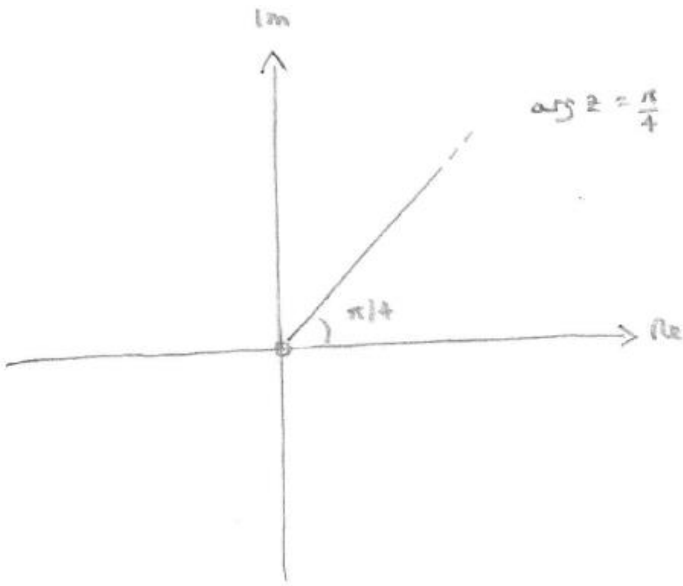
$\Rightarrow (x+\frac{4}{3})^2+(y+\frac{1}{3})^2=\frac{8}{9}$

ie a circle centre $\left(-\frac{4}{3}, -\frac{1}{3}\right)$, radius $\frac{2\sqrt{2}}{3}$



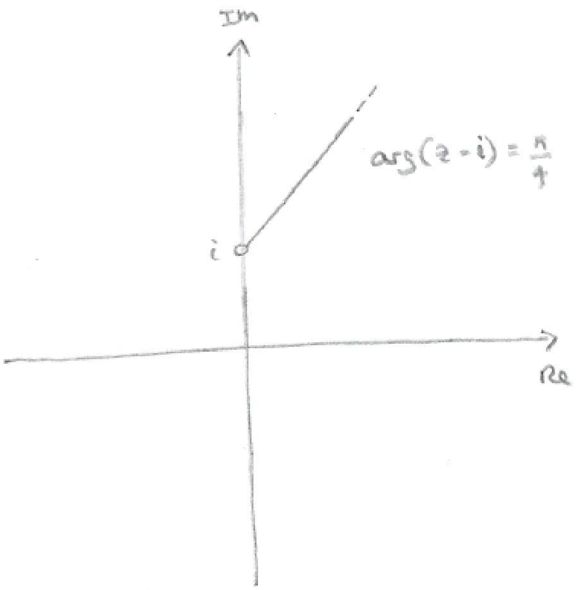

## (19) Loci involving arguments

Example: $\arg z = \frac{\pi}{4}$



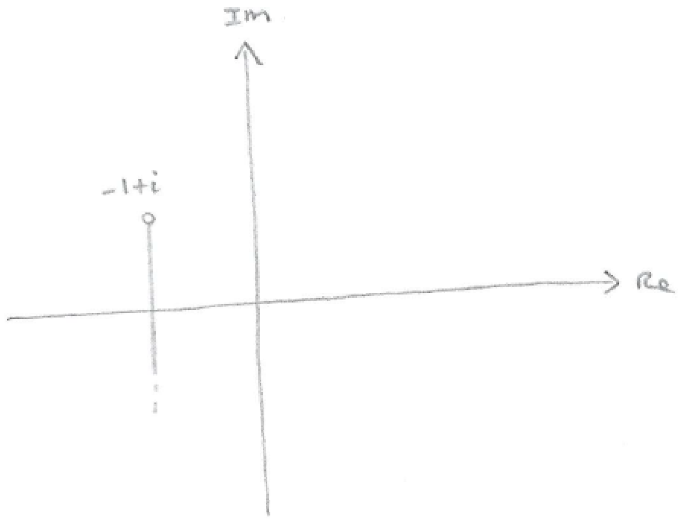

Note: The Origin is excluded, as $arg(0)$ is undefined

**Example:** $\arg(z - i) = \frac{\pi}{4}$

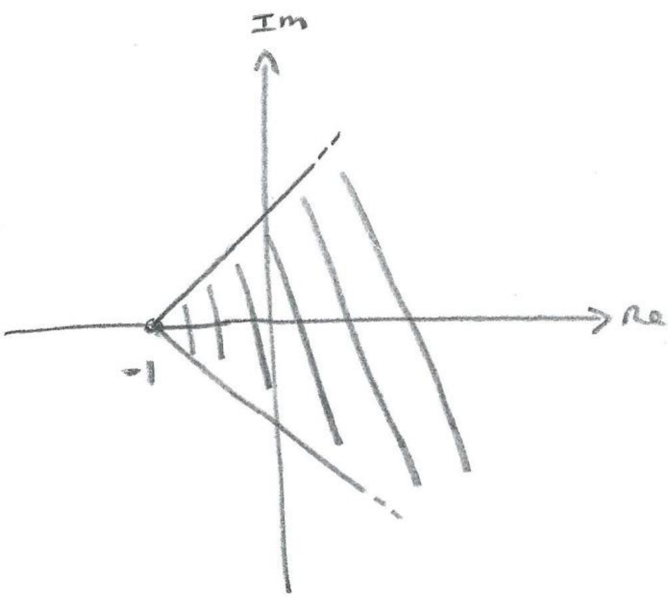



**Exercise:** Draw the locus of $\arg(z + 1 - i) = -\frac{\pi}{2}$

Rewriting as $\arg(z-[-1+i])=-\frac{\pi}{2}$:

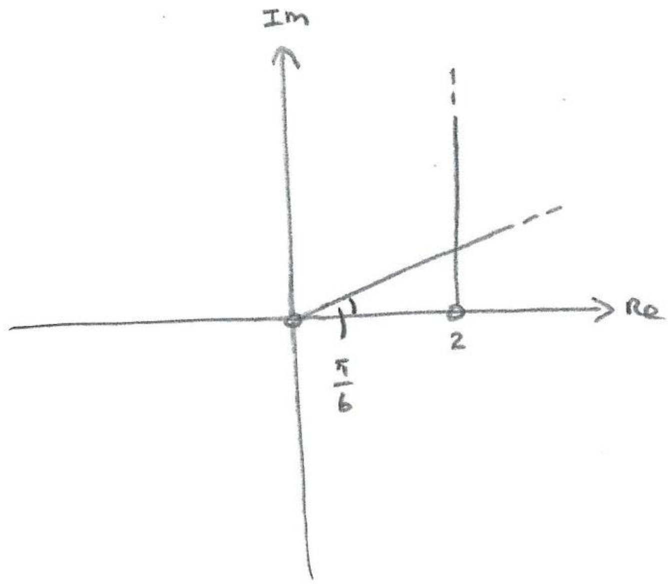



**Exercise**: Show in an Argand diagram the set of points satisfying the inequality $-\frac{\pi}{4}\le\arg(z+1)\le\frac{\pi}{4}$




**Example**: Solve the simultaneous equations:

$\arg(z-2)=\frac{\pi}{2}$ and $\arg z=\frac{\pi}{6}$

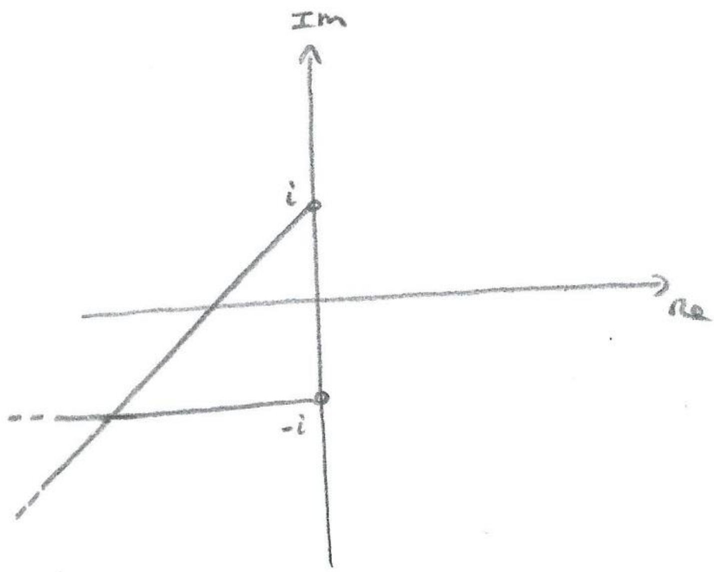



$\Rightarrow z = 2 + 2\tan(\frac{\pi}{6})i$

$= 2 + \frac{2}{\sqrt{3}}i$ or $2 + \frac{2\sqrt{3}}{3}i$

**Exercise**: Solve the simultaneous equations:

$\arg(z + i) = \pi$ and $\arg(z - i) = \frac{-3\pi}{4}$



$\Rightarrow z = -2 - i$

## (20) De Moivre's Theorem

The theorem states that, if $z = cos\theta + isin\theta$, then

$z^n = cos(n\theta) + isin(n\theta)$ , where $n$ can be fractional and/or negative

When $n$ is a positive integer, this follows from the result established earlier that, where $z_1 = r_1(cos\theta + isin\theta)$ and

$z_2 = r_2(cos\phi + isin\phi)$, then

$z_1 z_2 = r_1 r_2\{cos(\theta + \phi) + isin(\theta + \phi)\}$

Putting $z = z_1 = z_2$ gives $z^2 = cos(2\theta) + isin(2\theta)$, and this can be extended to higher integers by the same method.

**Exercise**: Express $(1 - i)^6$ in the form $x + iy$

**Solution**

First of all, express $z = 1 - i$ in modulus-argument form:

By considering the Argand diagram, $|z| = \sqrt{2}$ & $\arg(z) = -\frac{\pi}{4}$

So $z = \sqrt{2}(\cos\left(-\frac{\pi}{4}\right) + isin\left(-\frac{\pi}{4}\right))$

Then, by de Moivre's theorem,

$$z^6 = \left(\sqrt{2}\right)^6 \left(\cos\left(-\frac{6\pi}{4}\right) + isin(-\frac{6\pi}{4})\right)$$

$$= 8\left(\cos\left(-\frac{3\pi}{2}\right) + isin\left(-\frac{3\pi}{2}\right)\right)$$

$$= 8\left(\cos\left(\frac{\pi}{2}\right) + isin\left(\frac{\pi}{2}\right)\right) = 8i$$

When $n$ is a negative integer:

Let $n = -k$

Then $(cos\theta + isin\theta)^n = \frac{1}{(cos\theta+isin\theta)^k} = \frac{1}{cosk\theta+isink\theta}$

$= \frac{1}{cosk\theta+isink\theta} \cdot \frac{cosk\theta-isink\theta}{cosk\theta-isink\theta} = \frac{\cos(-k\theta)+isin(-k\theta)}{cos^2k\theta+sin^2k\theta}$

$= \cos(n\theta) + isin(n\theta)$

**Results following from de Moivre's theorem**

(i) $(cos\theta - isin\theta)^n = (\cos(-\theta) + isin(-\theta))^n$

$= \cos(-n\theta) + isin(-n\theta) = \cos(n\theta) - isin(n\theta)$

(ii) If $\mathrm{z} = cos\theta + isin\theta$,

then $z^{-1} = \cos(-\theta) + isin(-\theta) = cos\theta - isin\theta = z^*$

(but note that $z^{-1} = z^*$ only when $|z| = 1$; $zz^* = |z|^2$ also gives this result)

(iii) For general $z = r(cos\theta + isin\theta)$, $z^{-1} = \frac{1}{r}\,(cos\theta - isin\theta)$

$= \frac{1}{r} \cdot \frac{z^*}{r} = \frac{z^*}{|z|^2}$

De Moivre's theorem can also be shown to be true for fractional $n$.

(21) Using de Moivre's Theorem to establish Trig. identities: Multiple angle formulae

**Example**: Show that $cos2\theta = cos^2\theta - sin^2\theta$

$cos2\theta = Re\{cos2\theta + isin2\theta\} = Re\{(cos\theta + isin\theta)^2\}$

$= Re\{cos^2\theta + 2icos\theta sin\theta - sin^2\theta\}$

$= cos^2\theta - sin^2\theta$

(and similarly $sin2\theta = 2sin\theta cos\theta$)

**Exercise**: Find an expression for $sin3\theta$ in terms of powers of $sin\theta$ and/or $cos\theta$

**Solution**

$sin3\theta = Im(cos3\theta + isin3\theta)$

$cos3\theta + isin3\theta = (cos\theta + isin\theta)^3$

$= cos^3\theta + 3cos^2\theta(isin\theta) + 3cos\theta(isin\theta)^2 + (isin\theta)^3$

Hence $sin3\theta = 3cos^2\theta(sin\theta) - sin^3\theta$

$= 3(1 - sin^2\theta)(sin\theta) - sin^3\theta$

$= 3sin\theta - 4sin^3\theta$

## (22) Powers of Sines and Cosines

### Powers of Cosines

To find $cos^2\theta$ in terms of $cos2\theta$:

Starting point: $cos\theta = \frac{1}{2}(z + z^{-1})$,

where z $= cos\theta + isin\theta$ and $z^{-1} = cos\theta - isin\theta$

Then $cos^2\theta = \frac{1}{4}(z + z^{-1})^2 = \frac{1}{4}(z^2 + 2 + z^{-2})$

Now $z^2 + z^{-2} = (cos2\theta + isin2\theta) + (cos2\theta - isin2\theta) = 2cos2\theta$

Hence $cos^2\theta = \frac{1}{4}(2 + 2cos2\theta) = \frac{1}{2}(1 + cos2\theta)$

**Exercise**: Show that $cos^3\theta = \frac{1}{4}(cos3\theta + 3cos\theta)$

**Solution**

$cos\theta = \frac{1}{2}(z + z^{-1})$

where z $= cos\theta + isin\theta$   and  $z^{-1} = cos\theta - isin\theta$

So  $cos^3\theta = \frac{1}{8}(z + z^{-1})^3 = \frac{1}{8}\ (z^3 + 3z + 3z^{-1} + z^{-3})$

$= \frac{1}{8}\{3(z + z^{-1}) + (z^3 + z^{-3})\}$

$= \frac{1}{8}\{3(2cos\theta) + (2cos3\theta)\}$

$= \frac{1}{4}(cos3\theta + 3cos\theta)$

**Powers of Sines**

$isin\theta = \frac{1}{2}(z - z^{-1})$,

where z $= cos\theta + isin\theta$   and  $z^{-1} = cos\theta - isin\theta$

So  $-isin^3\theta = \frac{1}{8}(z - z^{-1})^3$   (1)

**Exercise**: Find an expression for $sin^3\theta$

**Solution**

(1) $\Rightarrow -8isin^3\theta = z^3 - 3z + 3z^{-1} - z^{-3}$

$= z^3 - z^{-3} - 3(z - z^{-1})$

$= 2isin(3\theta) - 3(2i)sin\theta$

Hence  $sin^3\theta = -\frac{1}{8}\ (2\sin(3\theta) - 6sin\theta) = \frac{1}{4}(3sin\theta - \sin(3\theta))$

**(23) Exponential form of complex number**

$z = r(cos\theta + isin\theta) = re^{i\theta}$

Justification (assuming knowledge of Maclaurin expansions of $e^x, cos\ x$ & $\sin x$):

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} \ldots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots)$$

$$= cos\theta + isin\theta$$

De Moivre's theorem is then simply: $\left(e^{i\theta}\right)^n = e^{i(n\theta)}$, as we would expect.

**(24) Roots of Complex Numbers**

Consider the equation $z^3 = cos\theta + isin\theta$

Then $z = cos\left(\frac{\theta}{3}\right) + isin\left(\frac{\theta}{3}\right)$ is a solution

But $cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right) + isin\left(\frac{\theta}{3} + \frac{2\pi}{3}\right)$ is a solution as well

and so is $cos\left(\frac{\theta}{3} + 2(\frac{2\pi}{3})\right) + isin\left(\frac{\theta}{3} + 2(\frac{2\pi}{3})\right)$

These are the solutions of $z = (cos\theta + isin\theta)^{1/3}$

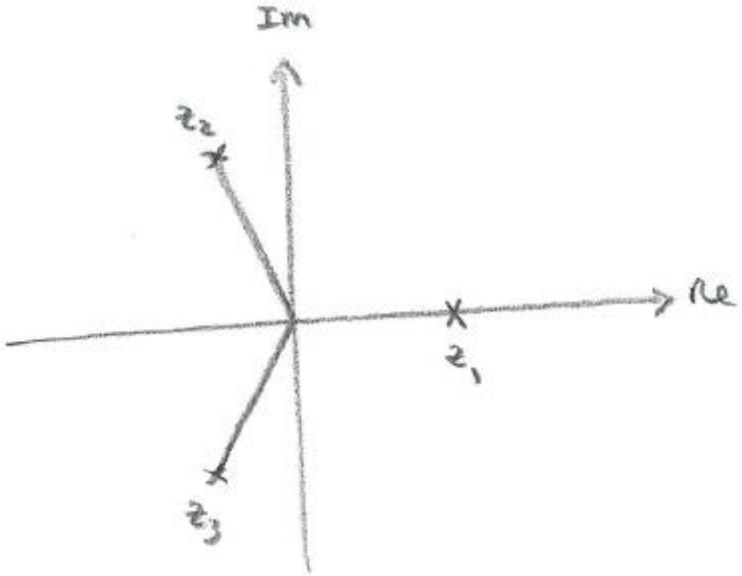
**Exercise**: When $\theta = 0$, express these 3 solutions in the form $a + bi$, and show them on the Argand diagram.

**Solution**

$$z_1 = cos\left(\frac{0}{3}\right) + isin\left(\frac{0}{3}\right) = 1$$

$$z_2 = cos\left(\frac{0}{3} + \frac{2\pi}{3}\right) + isin\left(\frac{0}{3} + \frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$z_3 = cos\left(\frac{0}{3} + 2(\frac{2\pi}{3})\right) + isin\left(\frac{0}{3} + 2(\frac{2\pi}{3})\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$




So there are 3 solutions of $z^3 = cos\theta + isin\theta$, spread evenly on a unit circle in the Argand diagram, starting at $\frac{\theta}{3}$. These are the 3 cube roots of $cos\theta + isin\theta$.

More generally, there will be $n$ roots of the equation

$z^n = r(cos\theta + isin\theta)$;

namely $z = r^{\frac{1}{n}}(\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + isin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right))$

for $k = 0,1, \dots, n-1$

Note that $\frac{\theta}{n} + \frac{2n\pi}{n} = \frac{\theta}{n} + 2\pi$, and so the root associated with $k = n$ is identical to that associated with $k = 0$

## (25) Relation between the roots of unity

**Example**: The 5 roots of $z^5 = 1$ (the "roots of unity") are

$cos\theta + isin\theta,$ where $\theta = \frac{2k\pi}{5}$, for $k = 0,1, \ldots ,4$

The 1st root after 1 is commonly denoted by $\omega$,

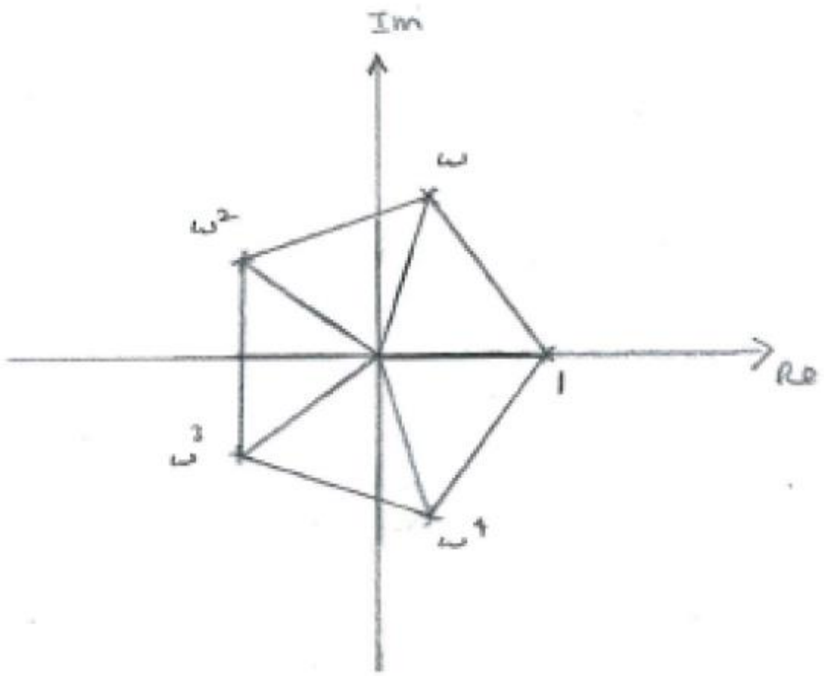
so that $\omega = cos\left(\frac{2\pi}{5}\right) + isin\left(\frac{2\pi}{5}\right)$

Then $\omega^2 = cos\left(\frac{4\pi}{5}\right) + isin\left(\frac{4\pi}{5}\right)$, by de Moivre's theorem.

In general, $\omega^k = cos\left(\frac{2k\pi}{5}\right) + isin\left(\frac{2k\pi}{5}\right)$,

and we can see that the 5 roots are: $1, \omega, \omega^2, \omega^3$ & $\omega^4$

These form the vertices of a polygon, as in the diagram below.




The following result will now be proved:

$1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$

**Approach 1** (algebraic)

This is a geometric series with common ratio $\omega$, and so

$LHS = \frac{\omega^5 - 1}{\omega - 1} = \frac{0}{\omega - 1}$ (as $\omega^5 = 1$) $= 0$ (as $\omega \neq 1$)

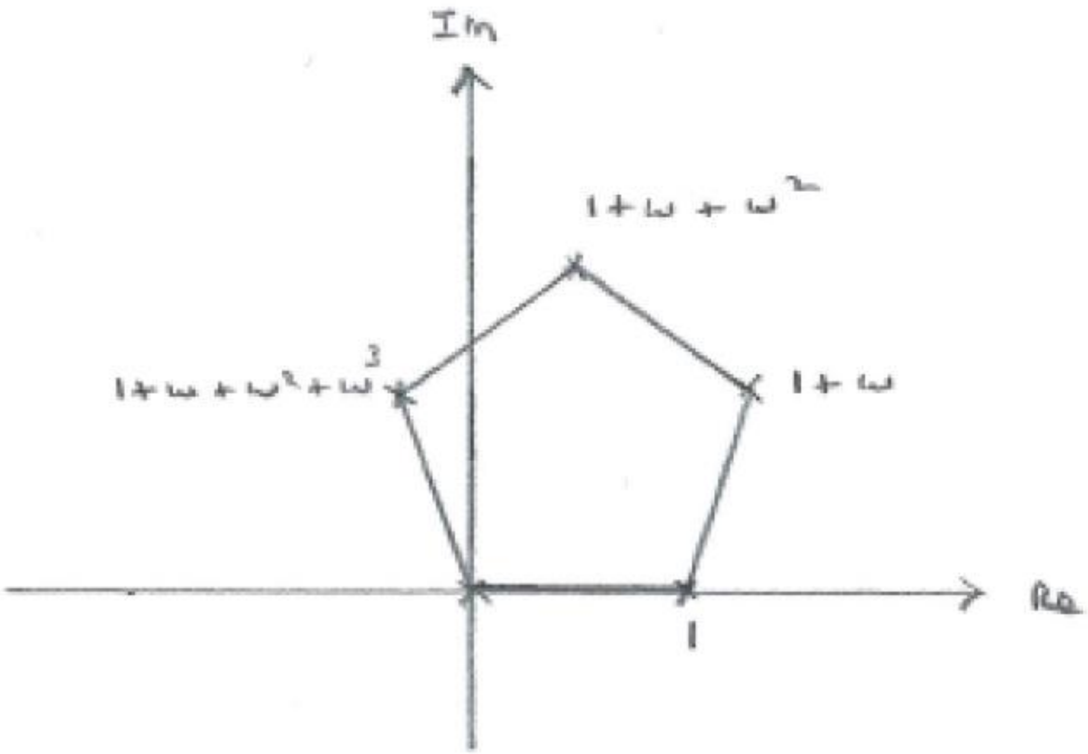
**Approach 2** (vectorial)

Treating complex numbers as vectors, $1 + \omega$ can be created as a vertex of the (new) polygon shown below. This then leads to

$1 + \omega + \omega^2$, and so on.

The 5 sides of the polygon are $1, \omega, \omega^2, \omega^3$ & $\omega^4$, in their vector form (each side has length 1, and the directions they make with the positive real axis are $0, \frac{2\pi}{5}, 2\left(\frac{2\pi}{5}\right), 3\left(\frac{2\pi}{5}\right), \ldots$ )

[Note that $1, \omega, \omega^2, \omega^3$ & $\omega^4$ were the **vertices** of the 1st polygon.]




From the diagram we see that the vector $1 + \omega + \omega^2 + \omega^3 + \omega^4$

is at the Origin; ie $1+\omega+\omega^2+\omega^3+\omega^4=0$

**Exercise**: If $1, \omega,\ \omega^2$ are the cube roots of 1, draw the polygon with vertices $1,\ 1+\omega,\ 1+\omega+\omega^2$

**Solution**