Complex Numbers - Part 1 (12 pages; 9/12/15)

(1) Consider $x^2 + 1 = 0 \quad \Rightarrow x = \sqrt{-1}$

Define $i = \sqrt{-1}$ so that $i^2 = -1$

For expressions involving i, apply the usual rules of algebra, and replace any occurrences of i^2 with -1

Examples

(i) $(1 + i)(2 + 3i) = 2 + 3i + 2i + 3i^2 = 2 + 5i - 3 = -1 + 5i$

(ii) $(2 + 3i)(2 - 3i) = 4 - (3i)^2 = 4 - 9i^2 = 4 - 9(-1)$

$= 4 + 9 = 13$

(iii) $i^7 = (i^4)(i^2)i = (1)(-1)i = -i$

(iv) To solve $z^2 + 16 = 0$: $z = \sqrt{-16} = \sqrt{16}\sqrt{-1} = 4i$

(v) $\frac{2}{i} = \frac{2i}{i^2} = \frac{2i}{-1} = -2i$

(The process in (v) is called 'rationalising the denominator'. Oddly enough, this is the same expression as is applied to $\frac{2}{\sqrt{3}}$, for example; even though in the case of $\frac{2}{i}$ we are making the denominator real, rather than rational!)

Note: j is sometimes used (especially by engineers) instead of i

(2) If $z = a + bi$, where a & b are real numbers, a is defined to be the **real part** of z, $Re(z)$ and b is the **imaginary part**, $Im(z)$.

An **imaginary** number is defined to be a number of the form bi

[For clarity, it is sometimes referred to as a 'pure imaginary' number.]

The definition of a complex number is a number of the form a $+ bi$ (where a and b can be zero).

Exercise: State whether the following are real, imaginary or complex (they could be more than one of these):

(a) 1 (b) $1 + 2i$ (c) $2i$ (d) 0

Solution

(a) real & complex

(b) complex

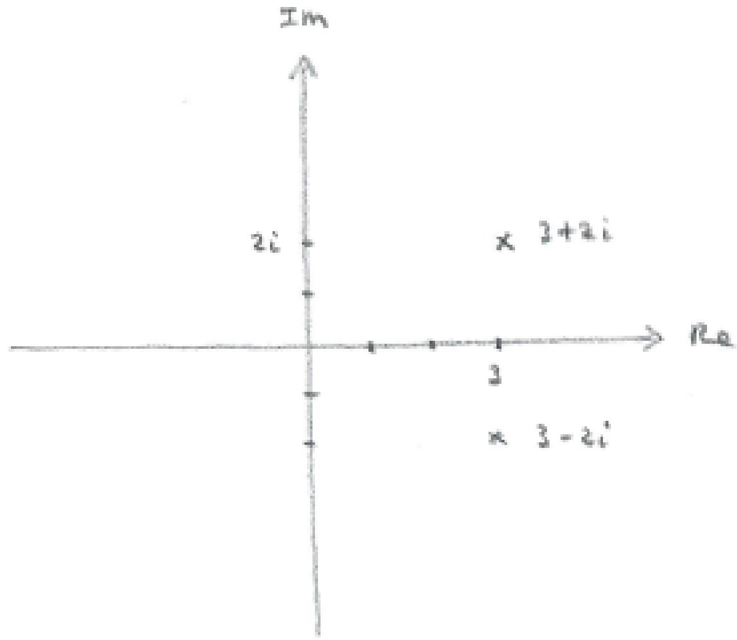
(c) imaginary & complex

(d) real, imaginary & complex!

For clarity, a number such as $1 + 2i$ is sometimes referred to as a 'non-real complex' number.

(3) Argand Diagram

Complex numbers can be treated in a very similar way to vectors, by representing $a + bi$ (where a & b are real) by the point (a, b) in the **'Argand diagram'** (see below).



The two axes are referred to as the **real** and **imaginary axes**.

Textbooks differ as to whether the imaginary axis should be labelled $i, 2i, 3i, \ldots$, or 1,2,3, ... The former is probably more common (and clearer).

We can see that a real number is any number on the real axis, whilst an imaginary number is any number on the imaginary axis (and a complex number is any number in the Argand diagram). This explains why the number 0 is real, imaginary and complex.

(4) Equating real and imaginary parts

Just as for vectors, if $a + bi = c + di$, it follows that

$a = c \quad and \quad b = d$

This is a (very) commonly-used and powerful technique in Complex Numbers.

(5) Quadratic equations

Consider $z^2 + z + 1 = 0$

$\Rightarrow z = \frac{-1\pm\sqrt{-3}}{2} = \frac{-1\pm\sqrt{-1}\sqrt{3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

If $z = a + bi$, then $z^* = a - bi$ is defined as the (complex) **conjugate** of z. [z^* is sometimes written as $\overline{z}$]

In the Argand diagram, z^* is the reflection in the real axis of z.

If the coefficients of a quadratic equation are real and one of the roots is a non-real complex number, then the other root will be its conjugate.

[The use of the letter z is always a hint that non-real complex roots are expected.]

Example: Find the quadratic equation with roots

$1 + i$ and $1 - i$

Method 1

$(z - [1 + i])(z - [1 - i)] = 0$ (A)

$\Rightarrow z^2 + z(-1 + i - 1 - i) + (1 - i^2) = 0$

$\Rightarrow z^2 - 2z + 2 = 0$

Or, from (A): $([z - 1] - i)([z - 1] + i) = 0$

$\Rightarrow (z - 1)^2 - i^2 = 0$

$\Rightarrow z^2 - 2z + 1 + 1 = 0$ etc

Method 2

Let the equation be $z^2 + bz + c = 0$

Using the fact that the sum of the roots of $az^2 + bz + c = 0$ is $-\frac{b}{a}$, whilst the product of the roots is $\frac{c}{a}$:

$(1 + i) + (1 - i) = -b$ and $(1 + i)(1 - i) = c$

So $b = -2$ and $c = 1^2 - i^2 = 1 + 1 = 2$

and the equation is $z^2 - 2z + 2 = 0$

(6) Division by Complex Numbers

Example: $(2 + 5i) \div (1 + 3i)$

Method 1

$$\frac{2+5i}{1+3i} = \frac{(2+5i)(1-3i)}{(1+3i)(1-3i)} = \frac{2+15-6i+5i}{1-(3i)^2} = \frac{17}{10} - \frac{i}{10}$$

Check: $\frac{1}{10}(17 - i)(1 + 3i) = \frac{1}{10}(17 + 3 - i + 51i) = 2 + 5i$

Method 2

Let $(2 + 5i) \div (1 + 3i) = a + bi$

Then $2 + 5i = (a + bi)(1 + 3i) = a + 3ai + bi - 3b$

Equating real & imaginary parts, $2 = a - 3b$ (1)

and $5 = 3a + b$ (2)

$3 \times (1)$: $6 = 3a - 9b$ (3)

$5 = 3a + b$ (2)

$(3) - (2) \Rightarrow 1 = -10b \Rightarrow b = -\frac{1}{10}$

$(1) \Rightarrow a = 2 + 3\left(-\frac{1}{10}\right) = \frac{20-3}{10} = \frac{17}{10}$

Exercise: Solve the equation $(2+i)z+3=0$

Solution

Method 1

$(2+i)z+3=0 \Rightarrow z=\frac{-3}{2+i}=\frac{-3(2-i)}{4+1}=\frac{-6+3i}{5}$

Method 2

Let $z=a+bi$

Then $(2+i)(a+bi)+3=0$

$\Rightarrow 2a-b+(a+2b)i+3=0$

Equating real parts: $2a-b=-3$ (1)

Equating imaginary parts: $a+2b=0$ (2)

Hence $2(-2b)-b=-3$ and $\therefore b=\frac{3}{5}$ and $a=-\frac{6}{5}$

Exercise Solve the equation $2z=iz^*+1$

Solution

Let $z=a+bi$

$\Rightarrow 2(a+bi)=i(a-bi)+1$

Equating real parts: $2a=b+1$

Equating imaginary parts: $2b=a$

$\Rightarrow b=\frac{1}{3}$ & $a=\frac{2}{3}$, so that $z=\frac{1}{3}(2+i)$

(7) Example: Find $\sqrt{i}$

Let $\sqrt{i}=a+bi$

Then $i=(a+bi)^2=a^2-b^2+2abi$

Note: Because $i = (a+bi)^2 \Rightarrow \pm\sqrt{i} = a + bi$ (rather than just $\sqrt{i} = a + bi$), we need to ensure that the eventual solution does satisfy $\sqrt{i} = a + bi$. In fact, it isn't an issue because if $a + bi$ is one solution we would expect $-(a + bi)$ to be a solution as well; ie $\sqrt{i} = -(a + bi)$, so that $-\sqrt{i} = a + bi$

Equating real & imaginary parts,

$2ab = 1$ (1) & $a^2 - b^2 = 0$ (2)

$(2) \Rightarrow b = \pm a$

Case 1: $\boldsymbol{b = a}$

$(1) \Rightarrow 2a^2 = 1$

$\Rightarrow a = b = \pm\frac{1}{\sqrt{2}}$

Case 2: $\boldsymbol{b = -a}$

$(1) \Rightarrow -2a^2 = 1$, which isn't possible

Hence solution is: $\sqrt{i} = \pm\frac{1}{\sqrt{2}}(1+i) = \pm\frac{\sqrt{2}}{2}(1+i)$

Check: $\frac{1}{2}(1+i)^2 = \frac{1}{2}(1 - 1 + 2i) = i$

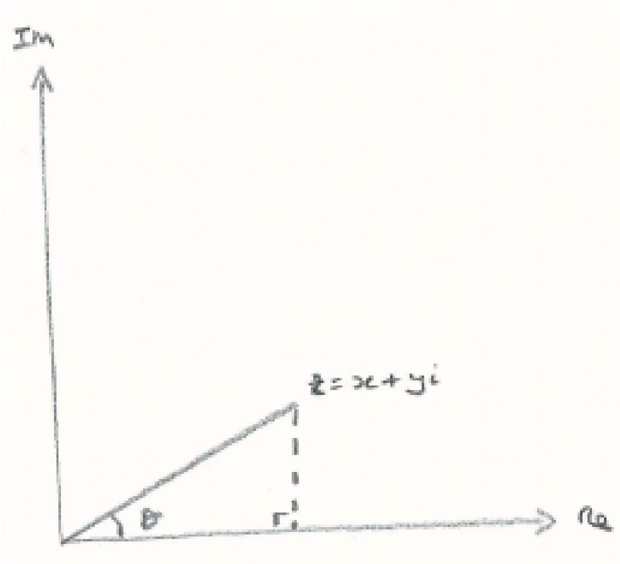
(8) Modulus

Referring to the diagram below, the modulus of z, denoted by $|z|$, is defined as the magnitude of z when viewed as a vector in the Argand diagram.

Thus $|z| = \sqrt{x^2 + y^2}$

Also, $(x + yi)(x - yi) = x^2 + y^2$,

so that $zz^* = |z|^2$



Note that the modulus is always positive; eg $|-2i| = 2$

(9) Argument

The argument of z, denoted by $\arg(z)$ [or just $\arg z$], is defined to be the angle that z makes with the positive real axis, when z is viewed as a vector in the Argand diagram.

So $\arg(z) = \theta$ in the above diagram.

The argument is usually measured in radians, and is usually restricted so that $-\pi < \arg(z) \leq \pi$

For example, if $\pi < \theta \leq 3\pi$, then $\arg(z) = \theta - 2\pi$,

and if $-3\pi < \theta \leq -\pi$, then $\arg(z) = \theta + 2\pi$

[An alternative convention that is sometimes used requires that $0 < \arg(z) \leq 2\pi$]

Note that $\arg(0)$ is not defined.

For $z = x + yi$, $tan\theta = \frac{y}{x}$ when z is in the 1st quadrant, and in this case $\arg(z) = arctan\left(\frac{y}{x}\right)$.

When z is not in the 1st quadrant, it is still the case that $tan\theta = \frac{y}{x}$ (by the definition of $sin\theta$ & $cos\theta$ for angles $\geq \frac{\pi}{2}$), but it may be necessary to add or subtract π from $arctan\left(\frac{y}{x}\right)$.

For example, $-3 - 2i$ is diametrically opposite $3 + 2i$, and

$\arg(-3 - 2i) = arctan\left(\frac{-2}{-3}\right) - \pi$

Also, $arctan(-1) = -\frac{\pi}{4}$, so that $\arg(-4 + 4i) = \arctan\left(\frac{4}{-4}\right) + \pi$

Sometimes the argument can be easily established by referring to the Argand diagram.

Examples

(i) $\arg(-3i) = -\frac{\pi}{2}$

(ii) $\arg(-1 + i) = \frac{3\pi}{4}$

(10) Polar (or modulus-argument) form

Let $r = |z| = \sqrt{x^2 + y^2}$

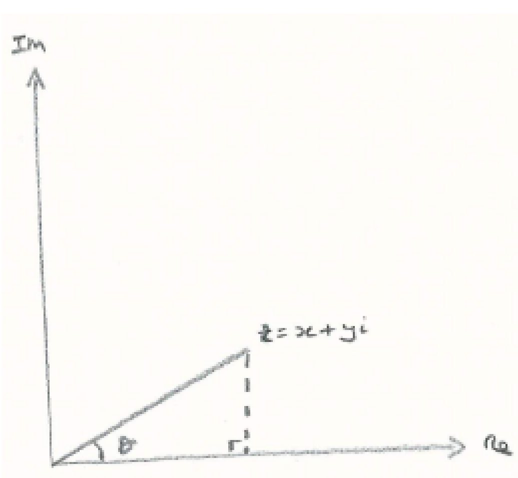
Then $x = rcos\theta$

and $y = rsin\theta$

Thus $z = r(cos\theta + isin\theta)$

This is the **polar** or **modulus-argument** form;

sometimes written as (r, θ) or (informally) as $rcis\theta$



Examples

(i) $\text{z} = 1 + \sqrt{3}\, i$

$|z| = \sqrt{1+3} = 2$

$\arg(z) = \arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$ (no adjustment is necessary, since z is in the 1st quadrant)

So $z = 2(cos\left(\frac{\pi}{3}\right) + isin\left(\frac{\pi}{3}\right)$

Alternatively, having found the modulus, we could write

$z = 2(\frac{1}{2} + \frac{\sqrt{3}}{2} i)$, and then recognise the angle from

$cos\theta = \frac{1}{2}$ & $sin\theta = \frac{\sqrt{3}}{2}$

(ii) $z = -2$

$|z| = 2$ & $\arg(z) = \pi$,

so $z = 2(\cos(\pi) + isin(\pi))$

(11) Product of two complex numbers

Let $z_1 = r_1(cos\theta + isin\theta)$ and $z_2 = r_2(cos\phi + isin\phi)$

Then $z_1 z_2 = r_1 r_2(cos\theta + isin\theta)(cos\phi + isin\phi)$

$= r_1 r_2\{(cos\theta cos\phi - sin\theta sin\phi) + i(sin\theta cos\phi + cos\theta sin\phi)\}$

$= r_1 r_2\{cos(\theta + \phi) + isin(\theta + \phi)\}$

(applying the compound angle formulae)

So $|z_1 z_2| = |z_1|.|z_2|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

Thus when z_1 is multiplied by z_2 , there are two effects:

The modulus of z_1 is multiplied by the modulus of z_2, and there is a rotation of $\phi\ rad$ anti-clockwise.

Note: we have to subtract 2π from $\theta + \phi$ if it exceeds π

(12) Dividing by a complex number

If $z_1 = r_1(cos\theta + isin\theta)$ and $z_2 = r_2(cos\phi + isin\phi)$,

$$\frac{z_1}{z_2} = \frac{r_1(cos\theta+isin\theta)}{r_2(cos\phi+isin\phi)}$$

$$= \frac{r_1(cos\theta+isin\theta)(cos\phi-isin\phi)}{r_2(cos\phi+isin\phi)(cos\phi-isin\phi)}$$

$$= \frac{r_1(cos\theta+isin\theta)(cos\phi-isin\phi)}{r_2(cos^2\phi+sin^2\phi)}$$

$$= \frac{r_1}{r_2}\{(cos\theta cos\phi + sin\theta sin\phi) + i(sin\theta cos\phi - cos\theta sin\phi)\}$$

$$= \frac{r_1}{r_2}\{cos(\theta - \phi) + isin(\theta - \phi)\}$$

So when z_1 is divided by z_2 :

The modulus of z_1 is divided by the modulus of z_2, and there is a rotation of $\phi\ rad$ clockwise.

Note: Add $2\pi\ to\ \theta - \phi$ if it is less than $-\pi$

(13) Exercises

(i) Use the modulus-argument form to establish the relation between z and iz on the Argand diagram.

Solution

Let $z = r(cos\theta + isin\theta)$ and write $i = cos\left(\frac{\pi}{2}\right) + isin\left(\frac{\pi}{2}\right)$

Then $iz = r\{cos\left(\theta + \frac{\pi}{2}\right) + isin\left(\theta + \frac{\pi}{2}\right)$

Thus iz is obtained from z by a rotation of $\frac{\pi}{2}$ radians (anti-clockwise) about the Origin.

(ii) Use the modulus-argument form to demonstrate that

$zz^* = |z|^2$

Solution

Let $z = r(cos\theta + isin\theta)$, so that

$z^* = r(cos\theta - isin\theta) = r(\cos(-\theta) + isin(-\theta))$

and $zz^* = r^2\{\cos(\theta - \theta) + isin(\theta - \theta)\}$

$= |z|^2(1)$

Alternatively:

$zz^* = r(cos\theta + isin\theta)r(cos\theta - isin\theta)$

$= r^2(cos^2\theta - (isin\theta)^2)$

$= r^2(cos^2\theta + sin^2\theta)$

$= |z|^2$