Complex Numbers - Exercises (Part 2) (Sol'ns)

(5 pages; 13/2/16)

(1) Find $\arg\{-sin\left(\frac{\pi}{3}\right) + icos\left(\frac{\pi}{3}\right)\}$, other than by just plotting the point in the Argand diagram.

Solution

Approach 1

$-sin\left(\frac{\pi}{3}\right) + icos\left(\frac{\pi}{3}\right) = sin\left(-\frac{\pi}{3}\right) + icos\left(-\frac{\pi}{3}\right)$

[note that it helps to keep the angle the same in both terms]

$= cos\left(\frac{\pi}{2} - [-\frac{\pi}{3}]\right) + isin\left(\frac{\pi}{2} - [-\frac{\pi}{3}]\right) = \cos\left(\frac{5\pi}{6}\right) + isin(\frac{5\pi}{6})$

So $\arg\{-sin\left(\frac{\pi}{3}\right) + icos\left(\frac{\pi}{3}\right)\} = \frac{5\pi}{6}$

Approach 2

$-sin\left(\frac{\pi}{3}\right) + icos\left(\frac{\pi}{3}\right) = -\cos\left(\frac{\pi}{2} - \frac{\pi}{3}\right) + isin\left(\frac{\pi}{2} - \frac{\pi}{3}\right)$

$= -cos\left(\frac{\pi}{6}\right) + isin\left(\frac{\pi}{6}\right) = -\{cos\left(\frac{\pi}{6}\right) - isin\left(\frac{\pi}{6}\right)\}$

Then $arg\left\{cos\left(\frac{\pi}{6}\right) - isin\left(\frac{\pi}{6}\right)\right\} = -\frac{\pi}{6}$

[as $cos\left(\frac{\pi}{6}\right) - isin\left(\frac{\pi}{6}\right)$ is the conjugate of $cos\left(\frac{\pi}{6}\right) + isin\left(\frac{\pi}{6}\right)$;

also $cos\left(\frac{\pi}{6}\right) - isin\left(\frac{\pi}{6}\right) = cos\left(-\frac{\pi}{6}\right) + isin\left(-\frac{\pi}{6}\right)$],

and so $\arg\left[-\left\{cos\left(\frac{\pi}{6}\right) - isin\left(\frac{\pi}{6}\right)\right\}\right] = -\frac{\pi}{6} + \pi = \frac{5\pi}{6}$

[since multiplication by -1 is a rotation by π in the Argand diagram]

(2) Find the mod and arg of $e^{\frac{7\pi i}{10}} - e^{-\frac{9\pi i}{10}}$

Solution

Method 1

Write $z = e^{\frac{7\pi i}{10}} - e^{-\frac{9\pi i}{10}}$ in the form $e^{a\pi i}(e^{b\pi i} - e^{-b\pi i})$

So $a + b = \frac{7}{10}$ & $a - b = -\frac{9}{10}$

Then $a = -\frac{1}{10}$ & $b = \frac{8}{10}$

and $e^{\frac{7\pi i}{10}} - e^{-\frac{9\pi i}{10}} = e^{-\frac{\pi i}{10}}(e^{\frac{8\pi i}{10}} - e^{-\frac{8\pi i}{10}})$

$= e^{-\frac{\pi i}{10}}(2isin\left(\frac{4\pi}{5}\right))$

Then $|z| = \left|e^{-\frac{\pi i}{10}}\right| \left|2isin\left(\frac{4\pi}{5}\right)\right| = (1)(2sin\left(\frac{4\pi}{5}\right))$

$= 2\sin\left(\pi - \frac{4\pi}{5}\right) = 2\sin(\frac{\pi}{5})$

and $\arg(z) = \arg\left(e^{-\frac{\pi i}{10}}\right) + \arg(2isin\left(\frac{4\pi}{5}\right))$

$= -\frac{\pi}{10} + \frac{\pi}{2} = \frac{4\pi}{10} = \frac{2\pi}{5}$

Method 2

$e^{\frac{7\pi i}{10}} - e^{-\frac{9\pi i}{10}}$

$= \left(cos\left(\frac{7\pi}{10}\right) - cos\left(\frac{-9\pi}{10}\right)\right) + i\left(sin\left(\frac{7\pi}{10}\right) - sin\left(\frac{-9\pi}{10}\right)\right)$

$= -2sin\left(\frac{1}{2}(\frac{7\pi}{10} + \frac{-9\pi}{10})\right) sin\left(\frac{1}{2}(\frac{7\pi}{10} - \frac{-9\pi}{10})\right)$

$+2cos\left(\frac{1}{2}(\frac{7\pi}{10} + \frac{-9\pi}{10})\right) sin\left(\frac{1}{2}(\frac{7\pi}{10} - \frac{-9\pi}{10})\right)$

$= -2sin\left(-\frac{\pi}{10}\right) sin\left(\frac{8\pi}{10}\right) + 2icos\left(-\frac{\pi}{10}\right) sin\left(\frac{8\pi}{10}\right)$

$= 2sin\left(\frac{8\pi}{10}\right)\{sin\left(\frac{\pi}{10}\right) + icos\left(\frac{\pi}{10}\right)\}$

$= 2sin\left(\frac{4\pi}{5}\right)\{cos\left(\frac{\pi}{2} - \frac{\pi}{10}\right) + isin\left(\frac{\pi}{2} - \frac{\pi}{10}\right)\}$

$= 2sin\left(\frac{\pi}{5}\right)\{cos\left(\frac{4\pi}{10}\right) + isin\left(\frac{4\pi}{10}\right)\}$

$= 2sin\left(\frac{\pi}{5}\right)e^{\frac{2\pi i}{5}}$

So mod is $2sin\left(\frac{\pi}{5}\right)$ and arg is $\frac{2\pi}{5}$

(3) Solve the equation $(1 + i)z^2 + (8 - 2i)z - 5(1 + 3i) = 0$

Solution

Simplify, by dividing through by $1 + i$:

$\frac{8-2i}{1+i} = \frac{(8-2i)(1-i)}{2} = \frac{6-10i}{2} = 3 - 5i$

$\frac{-5(1+3i)}{1+i} = \frac{-5(1+3i)(1-i)}{2} = \frac{-5(4+2i)}{2} = -5(2 + i)$

So equation becomes $z^2 + (3 - 5i)z - 5(2 + i) = 0$

Then $z = \frac{-(3-5i)\pm\sqrt{(3-5i)^2+20(2+i)}}{2} = \frac{-3+5i\pm\sqrt{24-10i}}{2}$

Suppose that $24 - 10i = (a + bi)^2 = a^2 - b^2 + 2abi$

Then $24 = a^2 - b^2$ and $-10 = 2ab$ (1)

Hence $24 = a^2 - \left(-\frac{5}{a}\right)^2$ and so $24a^2 = a^4 - 25$

and $a^4 - 24a^2 - 25 = 0$

$\Rightarrow (a^2 - 25)(a^2 + 1) = 0 \Rightarrow a^2 = 25$ or -1

Thus $a = \pm 5$ (as a is real)

Then, from (1), when $a = 5,\ b = -1$ and when $a = -5,\ b = 1$

Check: $(5 - i)^2 = 24 - 10i$

Thus $\pm\sqrt{24-10i} = 5 - i$ or $-5 + i$

and hence $z = \frac{-3+5i\pm\sqrt{24-10i}}{2} = \frac{2+4i}{2}$ or $\frac{-8+6i}{2}$

ie $z = 1 + 2i$ or $-4 + 3i$

[This can be checked by expanding $(z - [1 + 2i])(z - [-4 + 3i])$]

(4) Find i^i in cartesian form (ie $x + yi$)

Solution

$i^i = \left(e^{i(\frac{\pi}{2}+2k\pi)}\right)^i = e^{-(\frac{\pi}{2}+2k\pi)}$ for $k \in \mathbb{Z}$

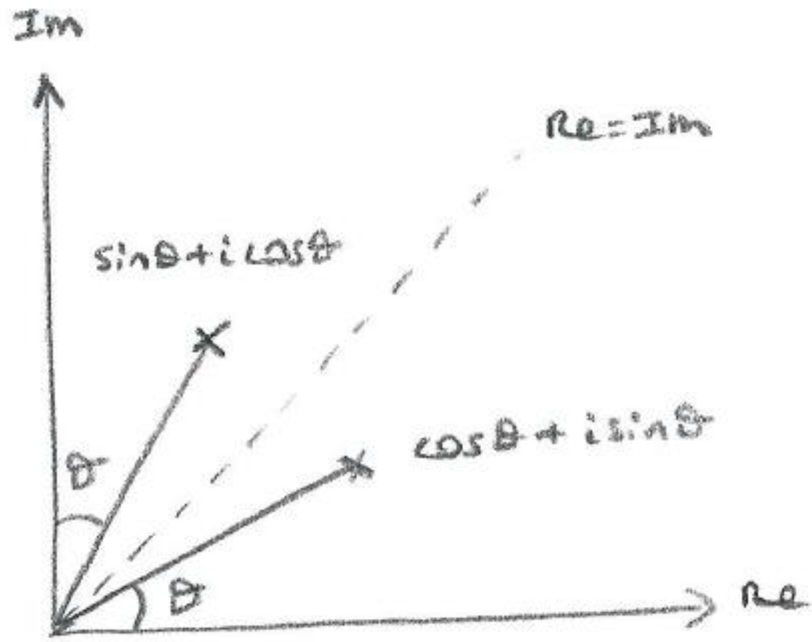
(ie i^i is a collection of real numbers)

(5) How are the complex numbers $cos\theta + isin\theta$ and $sin\theta + icos\theta$ related?

Solution

$sin\theta + icos\theta = \cos\left(\frac{\pi}{2} - \theta\right) + isin(\frac{\pi}{2} - \theta)$

As both complex numbers have a modulus of 1, $sin\theta + icos\theta$ is the reflection of $cos\theta + isin\theta$ in the line Re = Im (see diagram below).


Im
Re=Im
sinθ+icosθ
θ
cosθ+isinθ
θ
Re