Complex Numbers – Q4 (22/5/21)

Exam Boards

OCR : Pure Core (Year 2)

MEI: Core Pure (Year 2)

AQA: Pure (Year 2)

Edx: Core Pure (Year 2)

Find the solutions of $z^2 = i$ [4 marks]

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Solution

Method 1

$z^2 = i = \cos\left(\frac{\pi}{2}\right) + isin(\frac{\pi}{2})$ [1 mark]

By De Moivre's theorem, $z = \cos\left(\frac{\pi}{4}\right) + isin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(1+i)$

[1 mark]

or $z = \cos\left(\frac{\pi}{4} + \frac{(-2\pi)}{2}\right) + isin\left(\frac{\pi}{4} + \frac{(-2\pi)}{2}\right)$ [1 mark]

$= \cos\left(-\frac{3\pi}{4}\right) + isin(-\frac{3\pi}{4}) = -\frac{1}{\sqrt{2}}(1+i)$ [1 mark]

[Note that $\frac{\pi}{4} + \frac{(-2\pi)}{2}$ is chosen as the argument of the 2nd root, rather than $\frac{\pi}{4} + \frac{2\pi}{2}$, to avoid having to subtract 2π at the end.]

Method 2

Let $\sqrt{i} = a + bi$

Then $i = (a+bi)^2 = a^2 - b^2 + 2abi$

Equating real & imaginary parts,

$2ab = 1$ (1) & $a^2 - b^2 = 0$ (2)

$\Rightarrow a^2 - \left(\frac{1}{2a}\right)^2 = 0$

$\Rightarrow \left(a - \frac{1}{2a}\right)\left(a + \frac{1}{2a}\right) = 0$

$\Rightarrow$ either $a = \frac{1}{2a} \Rightarrow a^2 = \frac{1}{2} \Rightarrow a = \pm\frac{1}{\sqrt{2}}$

or $a = -\frac{1}{2a} \Rightarrow a^2 = -\frac{1}{2}$ (not possible, as a is real)

Then $a = +\frac{1}{\sqrt{2}} \Rightarrow b = \frac{1}{2a} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$, from (1)

and $a = -\frac{1}{\sqrt{2}} \Rightarrow b = -\frac{1}{\sqrt{2}}$

Thus $\sqrt{i} = \pm\frac{1}{\sqrt{2}}(1 + i)$

(This can be checked by squaring the RHS.)