Complex Numbers – Q11 (22/5/21)

Exam Boards

OCR : Pure Core (Year 2)

MEI: Core Pure (Year 2)

AQA: Pure (Year 2)

Edx: Core Pure (Year 2)

Find the modulus and argument of $e^{\frac{7\pi i}{10}} - e^{-\frac{9\pi i}{10}}$ [8 marks]

Find the modulus and argument of $e^{\frac{7\pi i}{10}} - e^{-\frac{9\pi i}{10}}$ [8 marks]

Solution

Method 1

Write $z = e^{\frac{7\pi i}{10}} - e^{-\frac{9\pi i}{10}}$ in the form $e^{a\pi i}(e^{b\pi i} - e^{-b\pi i})$ [1 mark]

So $a + b = \frac{7}{10}$ & $a - b = -\frac{9}{10}$ [1 mark]

Then $a = -\frac{1}{10}$ & $b = \frac{8}{10}$ [1 mark]

and $e^{\frac{7\pi i}{10}} - e^{-\frac{9\pi i}{10}} = e^{-\frac{\pi i}{10}}(e^{\frac{8\pi i}{10}} - e^{-\frac{8\pi i}{10}})$

$= e^{-\frac{\pi i}{10}}(2i\sin\left(\frac{4\pi}{5}\right))$ [1 mark]

Then $|z| = \left|e^{-\frac{\pi i}{10}}\right|\left|2i\sin\left(\frac{4\pi}{5}\right)\right| = (1)(2\sin\left(\frac{4\pi}{5}\right))$

$= 2\sin\left(\pi - \frac{4\pi}{5}\right) = 2\sin\left(\frac{\pi}{5}\right)$ [2 marks]

and $\arg(z) = \arg\left(e^{-\frac{\pi i}{10}}\right) + \arg\left(2i\sin\left(\frac{4\pi}{5}\right)\right)$

$= -\frac{\pi}{10} + \frac{\pi}{2} = \frac{4\pi}{10} = \frac{2\pi}{5}$ [2 marks]

Method 2

$e^{\frac{7\pi i}{10}} - e^{-\frac{9\pi i}{10}}$

$= \left(\cos\left(\frac{7\pi}{10}\right) - \cos\left(\frac{-9\pi}{10}\right)\right) + i\left(\sin\left(\frac{7\pi}{10}\right) - \sin\left(\frac{-9\pi}{10}\right)\right)$

$= -2\sin\left(\frac{1}{2}\left(\frac{7\pi}{10} + \frac{-9\pi}{10}\right)\right)\sin\left(\frac{1}{2}\left(\frac{7\pi}{10} - \frac{-9\pi}{10}\right)\right)$

$$+2cos\left(\frac{1}{2}\left(\frac{7\pi}{10}+\frac{-9\pi}{10}\right)\right)sin\left(\frac{1}{2}\left(\frac{7\pi}{10}-\frac{-9\pi}{10}\right)\right)$$

$$=-2sin\left(-\frac{\pi}{10}\right)sin\left(\frac{8\pi}{10}\right)+2icos\left(-\frac{\pi}{10}\right)sin\left(\frac{8\pi}{10}\right)$$

$$=2sin\left(\frac{8\pi}{10}\right)\left\{sin\left(\frac{\pi}{10}\right)+icos\left(\frac{\pi}{10}\right)\right\}$$

$$=2sin\left(\frac{4\pi}{5}\right)\left\{cos\left(\frac{\pi}{2}-\frac{\pi}{10}\right)+isin\left(\frac{\pi}{2}-\frac{\pi}{10}\right)\right\}$$

$$=2sin\left(\frac{\pi}{5}\right)\left\{cos\left(\frac{4\pi}{10}\right)+isin\left(\frac{4\pi}{10}\right)\right\}$$

$$=2sin\left(\frac{\pi}{5}\right)e^{\frac{2\pi i}{5}}$$

So mod is $2sin\left(\frac{\pi}{5}\right)$ and arg is $\frac{2\pi}{5}$