

Centre of Mass - Exercises (Solutions)

(9 pages; 7/8/17)

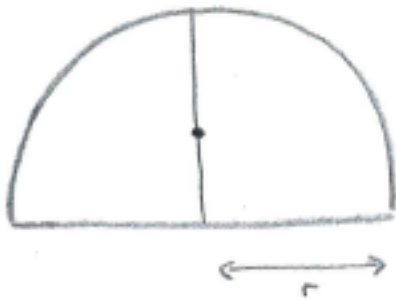
(1) Centre of Mass of Lamina by Integration

Find the centre of mass of a semi-circular lamina of radius r .

(a) by integrating wrt x

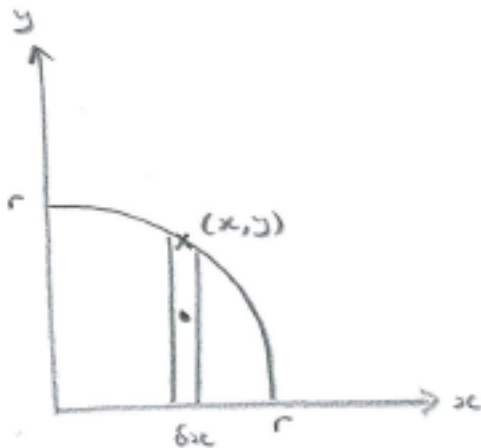
(b) by integrating wrt y

Solution



By symmetry, the CoM lies halfway across, as shown in the diagram. We just need to find the y component, \bar{y} . Again, by symmetry, \bar{y} for a quarter circle will equal \bar{y} for the semi-circle.

(a)



Considering vertical strips as shown in the diagram,

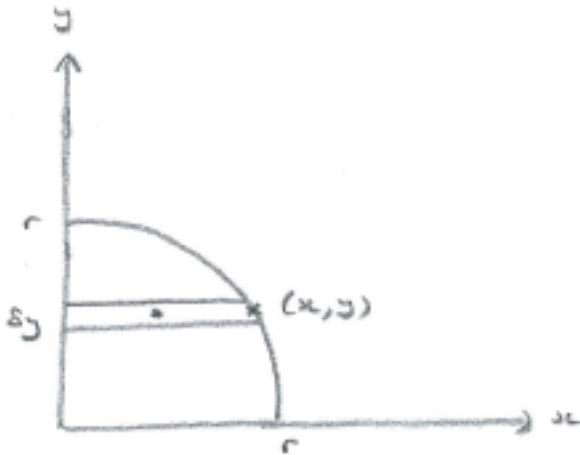
each strip has area $y\delta x$, and the total area of the quarter circle is

$$\frac{1}{4}\pi r^2. \quad \bar{y} \text{ for each strip is } \frac{y}{2}.$$

Weighting the CoM of each strip by its area and summing gives

$$\begin{aligned} \bar{y} &= \lim_{\delta x \rightarrow 0} \frac{1}{\left(\frac{\pi r^2}{4}\right)} \sum \frac{y}{2} \cdot y\delta x \\ &= \frac{1}{\left(\frac{\pi r^2}{4}\right)} \int_0^r \frac{y}{2} \cdot y dx = \frac{2}{\pi r^2} \int_0^r r^2 - x^2 dx, \quad \text{since } x^2 + y^2 = r^2 \\ &= \frac{2}{\pi r^2} \left[r^2 x - \frac{1}{3} x^3 \right]_0^r = \frac{2}{\pi r^2} \left(\frac{2r^3}{3} \right) = \frac{4r}{3\pi} \end{aligned}$$

(b)



Considering horizontal strips as shown in the diagram,

each strip has area $x\delta y$, and \bar{y} for each strip is y .

$$\text{Then } \bar{y} = \frac{1}{\left(\frac{\pi r^2}{4}\right)} \int_0^r y \cdot x dx = \frac{4}{\pi r^2} \int_0^r \sqrt{r^2 - x^2} \cdot x dx$$

As $\frac{d}{dx}(r^2 - x^2) = -2x$ and we have an x in the integrand,

we can make the substitution $u = r^2 - x^2$, so that $du = -2x dx$

$$\text{and } \bar{y} = \frac{4}{\pi r^2} \int_{r^2}^0 \sqrt{u} \cdot \left(-\frac{1}{2}\right) du = \frac{2}{\pi r^2} \int_0^{r^2} \sqrt{u} \cdot du$$

$$= \frac{2}{\pi r^2} \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^{r^2} = \frac{4r}{3\pi}, \text{ as before}$$

(2) Centre of mass of solid of revolution

The region between the curve $y = x^3 - x^2$ and the x -axis is rotated by 360° about the x -axis. Find the centre of mass of the solid of revolution obtained.

Solution

By symmetry, $\bar{y} = 0$

$y = x^3 - x^2 = x^2(x - 1)$ meets the x -axis at $x = 0$ & $x = 1$

$$V\bar{x} = \int_0^1 x(\pi y^2 dx), \text{ where } V = \int_0^1 \pi y^2 dx$$

[$\pi y^2 dx$ is the volume of the disc of width dx at position x]

$$\text{Thus } V = \pi \int_0^1 x^4(x - 1)^2 dx = \pi \int_0^1 x^6 - 2x^5 + x^4 dx$$

$$= \pi \left[\frac{1}{7} x^7 - \frac{2}{6} x^6 + \frac{1}{5} x^5 \right]_0^1$$

$$= \pi \left(\frac{1}{7} - \frac{2}{6} + \frac{1}{5} \right) = \frac{(30 - 70 + 42)\pi}{210} = \frac{\pi}{105} \text{ units}^3$$

$$\text{And } \bar{x} = 105 \int_0^1 x^7 - 2x^6 + x^5 dx$$

$$= 105 \left[\frac{1}{8} x^8 - \frac{2}{7} x^7 + \frac{1}{6} x^6 \right]_0^1$$

$$= 105 \left(\frac{1}{8} - \frac{2}{7} + \frac{1}{6} \right) = \frac{105(21-48+28)}{168}$$

$$= \frac{105}{168} = \frac{35}{56} = \frac{5}{8} = 0.625$$

Thus the centre of mass is (0.625, 0)

(3) Centre of mass of lamina

Find the centre of mass of the region between the curve

$y = x^3 - x^2$ and the x -axis.

Solution

$y = x^3 - x^2 = x^2(x - 1)$ meets the x -axis at $x = 0$ & $x = 1$

Note that the curve lies beneath the x -axis.

Total weight (signed area)

$$= \int_0^1 x^3 - x^2 dx = \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 \right]_0^1 = \left(\frac{1}{4} - \frac{1}{3} \right) = -\frac{1}{12}$$

$$-\frac{1}{12}\bar{x} = \int_0^1 x(x^3 - x^2 dx) = \left[\frac{1}{5}x^5 - \frac{1}{4}x^4 \right]_0^1 = \left(\frac{1}{5} - \frac{1}{4} \right) = -\frac{1}{20}$$

so that $\bar{x} = 0.6$

$$\text{And } -\frac{1}{12}\bar{y} = \int_0^1 \frac{y}{2}(x^3 - x^2 dx) = \frac{1}{2} \int_0^1 (x^3 - x^2)^2 dx$$

$$= \frac{1}{2} \int_0^1 x^6 + x^4 - 2x^5 dx$$

$$= \frac{1}{2} \left[\frac{1}{7}x^7 + \frac{1}{5}x^5 - \frac{2}{6}x^6 \right]_0^1$$

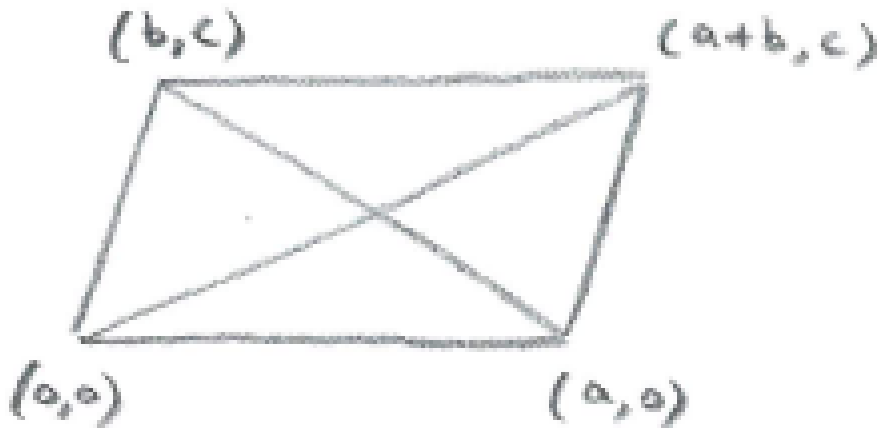
$$= \frac{1}{2} \left(\frac{1}{7} + \frac{1}{5} - \frac{2}{6} \right)$$

$$\text{so that } \bar{y} = \frac{-6(30+42-70)}{210} = -\frac{2}{35} = -0.0571 \text{ (3sf)}$$

Thus the centre of mass is (0.6, -0.0571)

(4) Show that the centre of mass of a parallelogram is at the intersection of the diagonals, by finding the centre of mass of two triangles, given the result that the diagonals bisect each other.

Solution



Let triangle 1 have corners

$(0,0)$, $(a,0)$ & (b,c)

(and triangle 2 be the other half of the parallelogram).

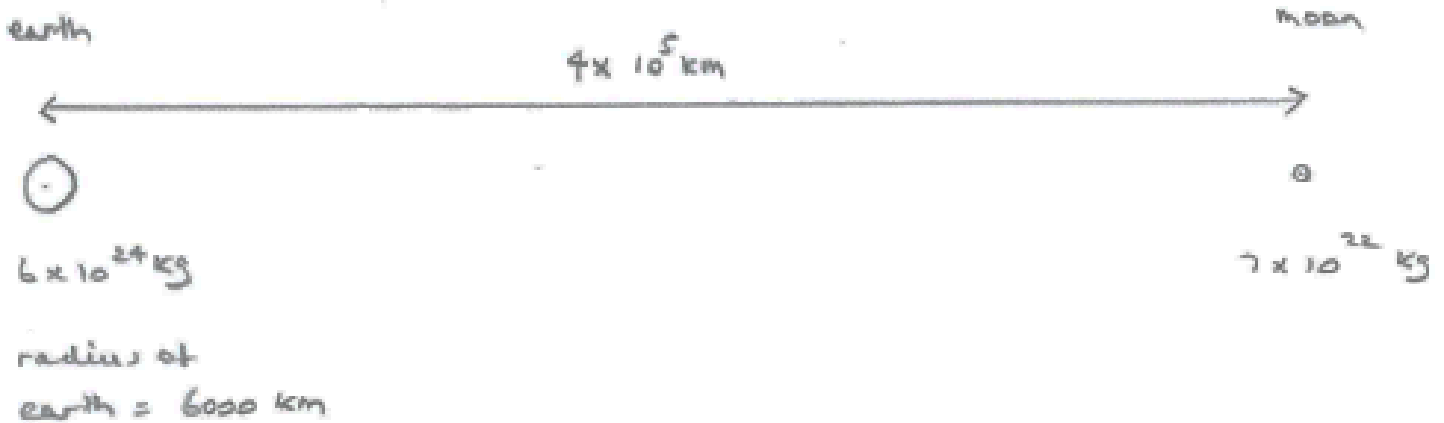
$$COM_1 = \begin{pmatrix} \frac{1}{3}(0 + a + b) \\ \frac{1}{3}(0 + 0 + c) \end{pmatrix} \quad \& \quad COM_2 = \begin{pmatrix} \frac{1}{3}(a + b + [a + b]) \\ \frac{1}{3}(0 + c + c) \end{pmatrix}$$

Then centre of mass of parallelogram = $\frac{1}{2} (COM_1 + COM_2)$

$$= \frac{1}{6} \begin{pmatrix} 3a + 3b \\ 3c \end{pmatrix} = \frac{1}{2} \begin{pmatrix} a + b \\ c \end{pmatrix}$$

ie the mid-point of the diagonal from $(0,0)$ to $(a+b,c)$

(5) Find the centre of mass of the Earth-Moon system



distance from earth to moon = $4 \times 10^5 \text{ km}$

radius of earth = 6000 km

mass of earth = $6 \times 10^{24} \text{ kg}$

mass of moon = $7 \times 10^{22} \text{ kg}$

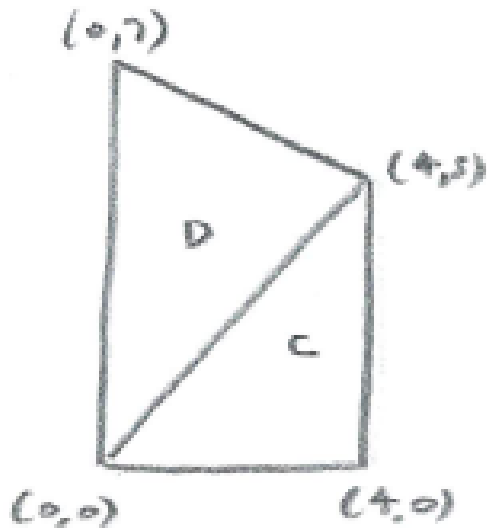
Solution

Taking the centre of the earth as the origin,

$$(6 \times 10^{24} + 7 \times 10^{22}) \bar{x} = 6 \times 10^{24} \times 0 + 7 \times 10^{22} \times 4 \times 10^5$$

$$\Rightarrow \bar{x} = \frac{28 \times 10^{27}}{607 \times 10^{22}} = 4600 \text{ km (2sf)}$$

(6) Find the centre of mass of the trapezium in the diagram, by dividing it up as shown.



Solution

$$\text{CoM of C: } \begin{pmatrix} \frac{1}{3}(0 + 4 + 4) \\ \frac{1}{3}(0 + 0 + 5) \end{pmatrix} = \begin{pmatrix} \frac{8}{3} \\ \frac{5}{3} \end{pmatrix}$$

$$\text{CoM of D: } \begin{pmatrix} \frac{1}{3}(0 + 4 + 0) \\ \frac{1}{3}(0 + 5 + 7) \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ 4 \end{pmatrix}$$

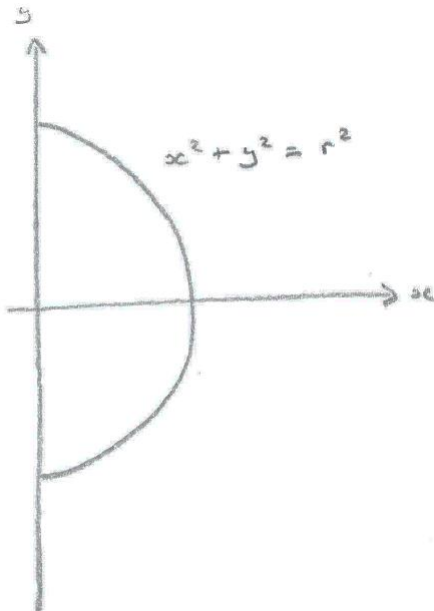
$$\text{Area of trapezium} = \frac{1}{2}(5 + 7)(4) = 24$$

$$\text{Area of C} = \frac{1}{2}(4)(5) = 10$$

$$\therefore \text{Area of D} = 14$$

$$\begin{aligned} \text{CoM of trapezium} &= \frac{1}{24} \begin{pmatrix} 10 \left(\frac{8}{3}\right) + 14\left(\frac{4}{3}\right) \\ 10 \left(\frac{5}{3}\right) + 14(4) \end{pmatrix} \\ &= \frac{1}{24} \begin{pmatrix} \frac{136}{3} \\ \frac{218}{3} \end{pmatrix} = \begin{pmatrix} \frac{17}{9} \\ \frac{109}{36} \end{pmatrix} \end{aligned}$$

(7) Find the centre of mass of the semi-circular lamina shown in the diagram.



Solution

By symmetry, we need only consider the top half :

$$\begin{aligned} \bar{x} &= \frac{1}{\frac{1}{4}\pi r^2} \int_0^r xy \, dx = \frac{4}{\pi r^2} \int_0^r x\sqrt{r^2 - x^2} \, dx \\ &= \frac{-2}{\pi r^2} \int_0^r (-2x)\sqrt{r^2 - x^2} \, dx \\ &= \frac{-2}{\pi r^2} \left[\frac{(r^2 - x^2)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^r \end{aligned}$$

$$= -\frac{4}{3\pi r^2}(0 - r^3)$$

$$= \frac{4r}{3\pi}$$