Boolean Algebra - Exercises (Solutions) (6 pages; 16/7/15)

Demonstrate the following, using the rules of Boolean algebra (see below)

(1) $a \vee \sim [(\sim b \vee a) \wedge b] = 1$

(2) $(a \vee b) \wedge (a \vee \sim b) = a$

(3) $\{a \vee (\sim b \wedge c)\} \wedge b \ = a \wedge b$

(4) $[a \wedge (b \vee c)] \vee \sim [\sim a \vee (b \wedge c)] = a$ [MEI, D2, June 2013, Q1]

(5) $(a \wedge b) \vee (\sim a \wedge \sim b) = (\sim a \vee b) \wedge (a \vee \sim b)$

[MEI, D2, June 2008, Q1]

(6) Show that $[(p \Rightarrow q) \wedge (\sim p \Rightarrow r)] \wedge \sim r \ \ \Rightarrow q$

[MEI, D2, June 2007, Q1]

(7) Given that the statements $(X \vee \sim Y) \Rightarrow Z$ and $\sim Z$ are both true, show that Y is true. [MEI, D2, June 2012, Q1]

(8) Show that $[\sim (X \vee \sim Y) \vee Z] \wedge (\sim Z) \Rightarrow Y$ [related to (7)]

Rules of Boolean algebra (See also "Logic: Implication")

Identity: $p \wedge 1 = p$; $p \vee 1 = 1$; $p \wedge 0 = 0$; $p \vee 0 = p$

Associative: $(p \vee q) \vee r = p \vee (q \vee r)$; $(p \wedge q) \wedge r = p \wedge (q \wedge r)$

Commutative: $p \vee q = q \vee p$; $p \wedge q = q \wedge p$

Complement: $\sim (\sim p) = p$; $p \vee (\sim p) = 1$; $p \wedge (\sim p) = 0$

Distributive: $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

De Morgan: $\sim (p \vee q) = (\sim p) \wedge (\sim q)$; $\sim (p \wedge q) = (\sim p) \vee (\sim q)$

Absorption: $p \wedge p = p$; $p \vee p = p$;

$p \wedge (p \vee q) = p$; $p \vee (p \wedge q) = p$

Solutions

Note: The = signs in the following solutions could all be written as either $\equiv$ or $\Leftrightarrow$

(1) $a \vee \sim [(\sim b \vee a) \wedge b] = 1$

LHS $= a \vee [\sim (\sim b \vee a) \vee \sim b]$ (De Morgan)

$= a \vee [(b \wedge \sim a) \vee \sim b]$ (De Morgan)

$= [a \vee (b \wedge \sim a)] \vee \sim b$ (Associative)

$= [(a \vee b) \wedge (a \vee \sim a)] \vee \sim b$ (Distributive)

$= [(a \vee b) \wedge 1] \vee \sim b$ (Complement)

$= (a \vee b) \vee \sim b$ (Identity)

$= a \vee (b \vee \sim b)$ (Associative)

$= a \vee 1$ (Complement)

$= 1$ (Identity)

(2) $(a \vee b) \wedge (a \vee \sim b) = a$

LHS $= [(a \vee b) \wedge a] \vee [(a \vee b) \wedge \sim b]$ (Distributive)

$= a \vee [(a \vee b) \wedge \sim b]$ (Absorption)

$= a \vee [(a \wedge \sim b) \vee (b \wedge \sim b)]$ (Distributive)

$= a \vee [(a \wedge \sim b) \vee 0]$ (Complement)

$= a \vee (a \wedge \sim b)$ (Identity)

$= a$ (Absorption)

(3) $\{a \vee (\sim b \wedge c)\} \wedge b = a \wedge b$

$LHS = \{(a \vee \sim b) \wedge (a \vee c)\} \wedge b$

$= \{(a \vee \sim b) \wedge b\} \wedge (a \vee c)$ (Associative & Commutative)

$= \{(a \wedge b) \vee (\sim b \wedge b)\} \wedge (a \vee c)$ (Distributive)

$= \{(a \wedge b) \vee 0\} \wedge (a \vee c)$ (Complement)

$= (a \wedge b) \wedge (a \vee c)$ (Identity)

$= \{a \wedge (a \vee c)\} \wedge b$ (Associative & Commutative)

$= a \wedge b$ (Absorption)

(4) $[a \wedge (b \vee c)] \vee \sim [\sim a \vee (b \wedge c)] = a$ (A)

First of all, $\sim [\sim a \vee (b \wedge c)] = a \wedge \sim (b \wedge c) = a \wedge (\sim b \vee \sim c)$

$= (a \wedge \sim b) \vee (a \wedge \sim c)$

Then aiming to obtain a collection of unions, which can then be rearranged by the Associativity rule:

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ [distributive rule]

So LHS of (A) $= (a \wedge b) \vee (a \wedge \sim b) \vee (a \wedge c) \vee (a \wedge \sim c)$

At this point, $= (a \wedge b) \vee (a \wedge \sim b)$ can be seen to equal a from a Venn diagram, and similarly for $(a \wedge c) \vee (a \wedge \sim c)$, giving the result of $a \vee a = a$

But if required to use Boolean algebra, then the distributive rule gives: $(a \wedge b) \vee (a \wedge \sim b) = a \wedge (b \vee \sim b) = a \wedge 1 = a$

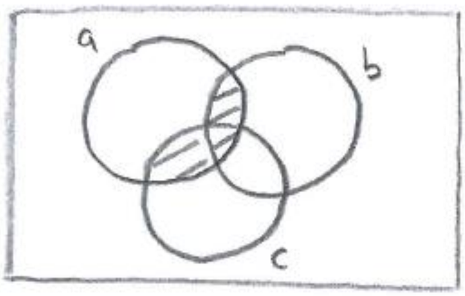
Alternatively, $[a \wedge (b \vee c)] \vee \sim [\sim a \vee (b \wedge c)] = a$ can be demonstrated using Venn Diagrams:

First of all, using Boolean algebra:

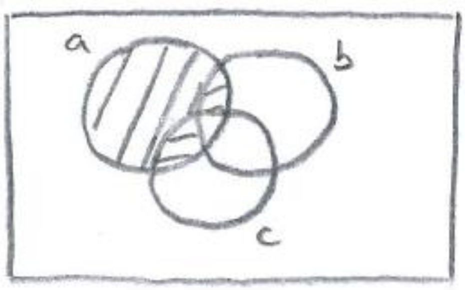
$\sim [\sim a \vee (b \wedge c)] = a \wedge \sim (b \wedge c) = a \wedge (\sim b \vee \sim c)$

$= (a \wedge \sim b) \vee (a \wedge \sim c)$, as before.

The LHS of (A) is then the union of the following 2 diagrams, and this can be seen to be a.



a ∧ (b ∨ c)



(a ∧ ~b) ∨ (a ∧ ~c)

[Also, the mark scheme to MEI, D2, June 2013, Q1 gives a solution using a truth table.]

(5) $(a \wedge b) \vee (\sim a \wedge \sim b) = (\sim a \vee b) \wedge (a \vee \sim b)$
[MEI, D2, June 2008, Q1]

By the Distributive rule, LHS$= [(a \wedge b) \vee \sim a] \wedge [(a \wedge b) \vee \sim b]$

$= [(a \vee \sim a) \wedge (b \vee \sim a)] \wedge [(a \vee \sim b) \wedge (b \vee \sim b)]$ (applying the Distributive rule to each of the square brackets)

$= (b \vee \sim a) \wedge (a \vee \sim b)$, which equals the RHS

[The mark scheme to MEI, D2, June 2008, Q1 gives a solution using a truth table.]

(6) Show that $[(p \Rightarrow q) \wedge (\sim p \Rightarrow r)] \wedge \sim r \quad \Rightarrow q$

Using the contrapositive of $\sim p \Rightarrow r$, $\sim r \Rightarrow p$

Then LHS $= (p \Rightarrow q) \wedge (\sim r \Rightarrow p) \wedge \sim r$ (A)

By 'modus ponens': $\sim r \wedge (\sim r \Rightarrow p) \Rightarrow p$

So (A) $\Rightarrow (p \Rightarrow q) \wedge p$

and, by 'modus ponens' again, $(p \Rightarrow q) \wedge p \Rightarrow q$, as required.

[See "Logic: Nested Tables - Exs" for alternative approach.]

(7) Given that the statements $(X \vee \sim Y) \Rightarrow Z$ and $\sim Z$ are both true, show that Y is true.

$(X \vee \sim Y) \Rightarrow Z \equiv \sim Z \Rightarrow \sim (X \vee \sim Y)$ [Contrapositive]

As $\sim Z$ is true, it follows that $\sim (X \vee \sim Y)$ is true

ie $\sim X \wedge Y$ is true [De Morgan]

so that Y is true

[It is also possible - though more time-consuming - to show that

$[(X \vee \sim Y) \Rightarrow Z] \wedge (\sim Z)$ is a subset of Y, by writing

$(X \vee \sim Y) \Rightarrow Z \equiv \sim (X \vee \sim Y) \vee Z$

This is a good exercise in itself though and appears as Exercise (8).]

(8) Show that $[\sim (X \vee \sim Y) \vee Z] \wedge (\sim Z) \Rightarrow Y$

$\text{LHS} \equiv [(\sim X \wedge Y) \vee Z] \wedge (\sim Z)$ [De Morgan]

$\equiv [(\sim X \vee Z) \wedge (Y \vee Z)] \wedge (\sim Z)$ [Distributive]

$\equiv (\sim X \vee Z) \wedge [(Y \wedge \sim Z) \vee (Z \wedge \sim Z)]$ [Associative & Distributive]

$\equiv (\sim X \vee Z) \wedge (Y \wedge \sim Z)$

$\equiv (\sim X \vee Z) \wedge Y \wedge \sim Z$, which is a subset of Y, and hence $\Rightarrow Y$