Arithmetic Sequences & Series - Exercises (Solutions)

(6 pages; 23/1/17)

[*/** indicates harder exercises]

(1) If teams A, B, C, D & E in some sporting competition have to play each other once, how many games are there in total?

Solution

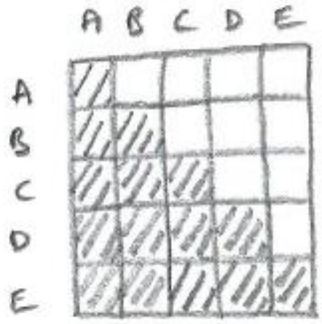
AvB, AvC, AvD, AvE   4 games

BvC, BvD, BvE   3 games

CvD, CvE 2 games

DvE 1 game

Total $= 1 + 2 + 3 + 4 = 10$ games



(2)(**) Extend (1) to find a formula for $1 + 2 + 3 + \cdots + n$

Ideas

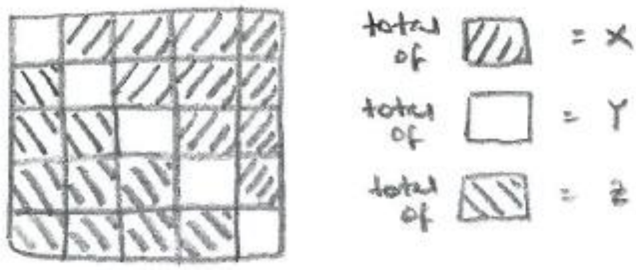
Consider the case $n = 4$

Use the diagram in (1)

Consider areas in the diagram

Give the areas letters

Can an equation be set up?



Solution

Divide the 5×5 square up into the areas X, Y & Z

Let the squares be of unit area.

Then $X = 1 + 2 + 3 + 4$,

$Y = 5$ & $Z = X$

So, for $n = 4$, $X + 5 + X = 25$

Generalising this, $2X + (n+1) = (n+1)^2$

$\Rightarrow 2X = (n+1)[(n+1) - 1] = (n+1)n$

$\Rightarrow X = \frac{n(n+1)}{2}$

(3) For each of the following arithmetic sequences, find an expression for a_k:

(a) in the form $a_k = p + q(k-1)$

(b) in the form $a_k = mk + c$

(c) in the form $a_k = a_{k-1} + t$; $a_1 = u$ $(k \geq 2)$

(where p, q, m, c, t & u are to be found)

(i) $4, 7, 10, 13, 16, \ldots$

Solution

(a) $a_k = 4 + 3(k-1)$

(b) $a_k = 3k + 1$

(c) $a_k = a_{k-1} + 3$; $a_1 = 4$ $(k \geq 2)$

(ii) $-2, -1, 0, 1, 2, \ldots$

Solution

(a) $a_k = -2 + (k-1)$

(b) $a_k = k - 3$

(c) $a_k = a_{k-1} + 1$; $a_1 = -2$ $(k \geq 2)$

(iii) $8, 6, 4, 2, 0, \ldots$

Solution

(a) $a_k = 8 - 2(k - 1)$

(b) $a_k = 10 - 2k$

(c) $a_k = a_{k-1} - 2$; $a_1 = 8$ $(k \geq 2)$

(4) If $a_3 = 7$ and $a_{10} = 42$ are terms in an arithmetic sequence, find an expression for a_k.

Solution

Let $a_k = mk + c$

Then $7 = 3m + c$ & $42 = 10m + c$

Hence $35 = 7m$, so that $m = 5$ & $c = -8$

ie $a_k = 5k - 8$

(5)(*) Find

(i) $\sum_{k=1}^{20}(2k + 3)$

Solution

We know how to find $\sum_{k=1}^{20} k$

Consider $\sum_{k=1}^{20} 2k$

It equals $2\sum_{k=1}^{20} k$

$\sum_{k=1}^{20}(2k + 3) = \left(\sum_{k=1}^{20} 2k\right) + \left(\sum_{k=1}^{20} 3\right)$

$\sum_{k=1}^{20} 3 = 3 + 3 + 3 + \cdots + 3 = (20)(3)$

So $\sum_{k=1}^{20}(2k + 3) = (2\sum_{k=1}^{20} k) + (20)(3)$

$= 2\left(\frac{20(20+1)}{2}\right) + 60$

$= 420 + 60 = 480$

(ii) $\sum_{k=1}^{40}(10 - 4k)$

Solution

$\sum_{k=1}^{40}(10 - 4k) = (10\sum_{k=1}^{40} 1\ ) - (4\sum_{k=1}^{40} k)$

$= 10(40) - 4\left(\frac{40(40+1)}{2}\right) = 400 - 3280 = -2880$

(6)(*) Solve the equation $\sum_{k=1}^{n}(100 - 5k) = 0$

Solution

$\sum_{k=1}^{n}(100 - 5k) = 0 \Rightarrow 100\sum_{k=1}^{n} 1 = 5\sum_{k=1}^{n} k$

$\Rightarrow 100n = 5\left(\frac{n(n+1)}{2}\right)$

$\Rightarrow 40n = n(n + 1) \Rightarrow n(n - 39) = 0$

Hence either $n = 0$ (not possible), or $n = 39$.

[Check: With $a = 100 - 5 = 95, d = -5, n = 39$,

$\frac{n}{2}[2a + (n - 1)d] = \frac{39}{2}[190 + 38(-5)] = 0$]

(7) For each of the arithmetic sequences in (5), find $\sum_{k=1}^{100} a_k$

Solution

(i) $4, 7, 10, 13, 16, \ldots$

$\sum_{k=1}^{100} a_k = \frac{100}{2}\left(2(4) + 3(100 - 1)\right) = 15250$

(ii) $-2, -1, 0, 1, 2, \ldots$

$\sum_{k=1}^{100} a_k = \frac{100}{2}\big(2(-2) + (100-1)\big) = 4750$

(iii) $8, 6, 4, 2, 0, \ldots$

$\sum_{k=1}^{100} a_k = \frac{100}{2}\big(2(8) + (-2)(100-1)\big) = -9100$

(8) If I pay £50 into a bank account, then £60 a year later, followed by £70 the following year, and so on, increasing by £10 each year,

(i) How long will it take for the amount I pay in each year to reach £200?

Solution

$50 + 10(k-1) = 200 \Rightarrow 10k = 160 \Rightarrow k = 16$

So, at the start of the 16th year I will be paying in £200.

(ii) How long will it take for the amount in the bank account to reach £1000?

Solution

Consider $\frac{n}{2}[(2(50) + 10(n-1)] = 1000$

$\Rightarrow n(90 + 10n) = 2000$

$\Rightarrow n^2 + 9n - 200 = 0$

$\Rightarrow n = \frac{-9 \pm \sqrt{81+800}}{2} = 10.3$ (as $n > 0$)

So, at the start of the 11th year (after paying in the amount due then) there will be over £1000 in the account.

(9)(*) For an arithmetic sequence with 1st term a and common difference d, show that the sum of the 1st n terms is

$\frac{n}{2}[2a+(n-1)d]$ by starting with $\sum_{k=1}^{n}[a+(k-1)d]$

Solution

$\sum_{k=1}^{n}[a+(k-1)d] = [(a-d)\sum_{k=1}^{n}1] + d\sum_{k=1}^{n}k$

$= (a-d)n + d\,.\frac{1}{2}\,n(n+1)$

$= \frac{n}{2}(2a-2d+dn+d) = \frac{n}{2}[2a+(n-1)d]$