Arc Lengths & Surface Areas of Revolution

(3 pages; 10/5/17)

(1) By Pythagoras, $(\delta s)^2 \approx (\delta x)^2 + (\delta y)^2$

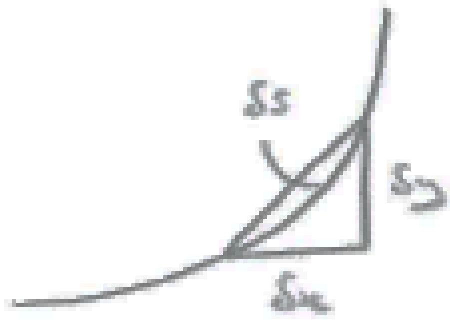
$\Rightarrow \left(\frac{\delta s}{\delta x}\right)^2 \approx 1 + \left(\frac{\delta y}{\delta x}\right)^2$

$\Rightarrow \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$ in the limit as $\delta s, \ \delta x \ \& \ \delta y \ \to 0$

Similarly $\left(\frac{ds}{dy}\right)^2 = \left(\frac{dx}{dy}\right)^2 + 1$

and $\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$

(for the case where x & y are expressed parametrically)



(2) Integrating to find an arc length

$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 \Rightarrow \ \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

and $\mathrm{s} = \int_{s_A}^{s_B} ds = \int_{x_A}^{x_B} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$\left(\frac{ds}{dy}\right)^2 = \left(\frac{dx}{dy}\right)^2 + 1 \Rightarrow \ \mathrm{s} = \int_{y_A}^{y_B} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

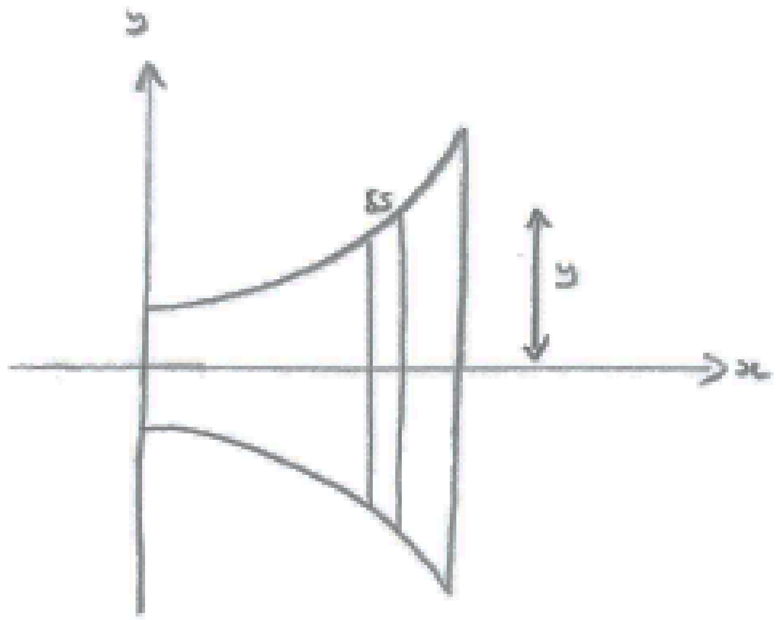
$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \Rightarrow \ s = \int_{t_A}^{t_B} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

(3) Example: Find the arc length of $y = coshx$ from $x = 0$ to $x = a$

$$s = \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^a \sqrt{1 + sinh^2 x}\, dx$$

$$= \int_0^a coshx\, dx = sinha - sinh0 = sinha$$

(4) Integrating to find the surface area of revolution



Surface area $= \lim_{\delta s \to 0} \sum (2\pi y)\delta s$

$$= \int_{s_A}^{s_B} 2\pi y\, ds$$

$$= \int_{x_A}^{x_B} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

or $\int_{t_A}^{t_B} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

and similarly about the y-axis (ie swapping roles of x & y)

Notes

(1) $\int_{y_A}^{y_B} 2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ is also possible for revolution about the x-axis

(2) Note that, for the volume of revolution, δs is approximated by δx (whereas for the surface area no such approximation is made). In the case of the volume however, δs is only the measurement at the edge of the infinitesimal disc (and is therefore of a small order), whereas for the surface area it is what we are integrating over.

(5) Example: Surface area of a hemisphere

Surface area $= \int_{t_A}^{t_B} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

$= \int_0^{\frac{\pi}{2}} 2\pi(rsint)\sqrt{(-rsint)^2+(rcost)^2}$ dt

$= 2\pi r\int_0^{\frac{\pi}{2}} rsint\,dt$

$= 2\pi r^2[-cost]_0^{\frac{\pi}{2}}$

$= 2\pi r^2$(0-(-1))

$= 2\pi r^2$