

Second Order Linear Differential Equations with Constant Coefficients (7 pages; 18/11/18)

(1) Example 1: $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = x^2$ (*)

where $y = 1$ and $\frac{dy}{dx} = 2$ when $x = 0$

It has been found that a function of the form $y = Ae^{\lambda x}$ will satisfy the **homogeneous** equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$.

[$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = x^2$ is referred to as a **non-homogeneous** equation, and its solution will follow shortly.]

Substituting this function into (*) gives:

$$\lambda^2 Ae^{\lambda x} - \lambda Ae^{\lambda x} - 6Ae^{\lambda x} = 0,$$

which leads to the **auxiliary equation** $\lambda^2 - \lambda - 6 = 0$ or $(\lambda + 2)(\lambda - 3) = 0$, which has roots $\lambda = -2$ & 3 .

The general solution of $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$ is then

$y = Ae^{-2x} + Be^{3x}$. (It can be verified that this satisfies (*), and it can be proved that it is the most general solution, by virtue of having two arbitrary constants.)

$y = Ae^{-2x} + Be^{3x}$ is referred to as the **complementary function** of non-homogeneous equation.

(2) We now find a single solution of the non-homogeneous equation; say $y = f(x)$ (with no arbitrary constants).

It can then be seen that $y = Ae^{-2x} + Be^{3x} + f(x)$ satisfies the non-homogeneous equation, and once again the presence of two arbitrary constants means that it is the general solution.

(3) The single solution is referred to as a **particular integral**. (Not to be confused with 'particular solution', which refers to a solution of a differential equation that satisfies specified initial or boundary conditions). Its form will depend on the right-hand side of the non-homogeneous equation, and a table of **trial functions** is given in Appendix 1.

For the present example, where the right-hand side is x^2 , the trial function is the quadratic function $y = ax^2 + bx + c$.

Substituting this into (*) gives

$$2a - (2ax + b) - 6(ax^2 + bx + c) = x^2$$

Equating the coefficients of powers of x :

$$x^2: -6a = 1; \quad x: -2a - 6b = 0; \quad \text{const. term: } 2a - b - 6c = 0$$

$$\text{leading to } a = -\frac{1}{6}, \quad b = \frac{1}{18}, \quad c = -\frac{7}{108}$$

The general solution of the non-homogeneous equation is therefore

$$y = Ae^{-2x} + Be^{3x} - \frac{x^2}{6} + \frac{x}{18} - \frac{7}{108}$$

As $y = 1$ and $\frac{dy}{dx} = 2$ when $x = 0$,

$$1 = A + B - \frac{7}{108}$$

$$\text{and } 2 = -2A + 3B + \frac{1}{18}$$

leading to $A = \frac{1}{4}$ and $B = \frac{22}{27}$

so that the particular solution is

$$y = \frac{1}{4}e^{-2x} + \frac{22}{27}e^{3x} - \frac{x^2}{6} + \frac{x}{18} - \frac{7}{108}$$

(4) Example 2: $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \cos 2x$ (*)

The auxiliary equation is $\lambda^2 - 6\lambda + 9 = 0$ or $(\lambda - 3)^2 = 0$, which has a repeated root of $\lambda = 3$.

We need a general solution with two arbitrary constants (for a 2nd order differential equation), and it can be shown that a suitable complementary function in this case is $y = (A + Bx)e^{3x}$ (see Appendix 2).

The particular integral in this case has been found to be of the form $y = a\sin 2x + b\cos 2x$.

Substituting into (*) gives

$$\begin{aligned} &(-4a\sin 2x - 4b\cos 2x) - 6(2a\cos 2x - 2b\sin 2x) \\ &+ 9(a\sin 2x + b\cos 2x) = \cos 2x \end{aligned}$$

Equating coefficients of $\sin 2x$ and $\cos 2x$:

$$-4a + 12b + 9a = 0 \quad \text{and} \quad -4b - 12a + 9b = 1$$

leading to $a = -\frac{12}{169}$ and $b = \frac{5}{169}$

The general solution is therefore

$$y = (A + Bx)e^{3x} - \frac{12\sin 2x}{169} + \frac{5\cos 2x}{169}$$

(5) Example 3: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = e^{3x}$ (*)

The auxiliary equation is $\lambda^2 - 2\lambda + 5 = 0$, which has

roots $\lambda = 1 \pm 2i$

The complementary function is then $y = e^x(Ae^{2ix} + Be^{-2ix})$

$$= e^x\{A\cos 2x + iA\sin 2x + B\cos(-2x) + iB\sin(-2x)\}$$

$$= e^x\{(A + B)\cos 2x + i(A - B)\sin 2x\}$$

$$= e^x(C\cos 2x + D\sin 2x)$$

[Note that C and D will be real if B is the complex conjugate of A . As only real values of C and D are acceptable, it follows that A and B are restricted to be complex conjugates.]

The particular integral in this case has been found to be of the form ae^{3x} .

Substituting into (*) gives $9ae^{3x} - 2(3ae^{3x}) + 5ae^{3x} = e^{3x}$,

so that $8a = 1$ and $a = \frac{1}{8}$.

The general solution is therefore

$$y = e^x(C\cos 2x + D\sin 2x) + \frac{e^{3x}}{8}$$

Note: Functions such as $C\cos 2x + D\sin 2x$ can also be written in the form $E\sin(2x + \alpha)$, for example.

(6) Example 4: $\frac{d^2y}{dx^2} + 9y = \sin 3x$ (*)

The auxiliary equation is $\lambda^2 + 9 = 0$, which has roots $\lambda = \pm 3i$

The complementary function is then $A\sin 3x + B\cos 3x$.

A particular integral of the form $asin3x + bcos3x$ won't work (as the left-hand side will equal 0). We therefore adopt a trial function of $x(asin3x + bcos3x)$.

[For this reason, it is a good idea to find the complementary function before the particular integral.]

For this trial function,

$$\frac{dy}{dx} = (asin3x + bcos3x) + x(3acos3x - 3bsin3x)$$

$$\text{and } \frac{d^2y}{dx^2} = 3acos3x - 3bsin3x + (3acos3x - 3bsin3x)$$

$$+x(-9asin3x - 9bcos3x)$$

$$= 6acos3x - 6bsin3x - 9x(asin3x + bcos3x)$$

Then, substituting into (*);

$$6acos3x - 6bsin3x - 9x(asin3x + bcos3x)$$

$$+9x(asin3x + bcos3x) = sin3x$$

$$\text{so that } 6acos3x - 6bsin3x = sin3x$$

Equating coefficients of $sin3x$ and $cos3x$:

$$a = 0 \text{ and } b = -\frac{1}{6}$$

The general solution is therefore

$$y = Asin3x + Bcos3x - \frac{1}{6}xcos3x$$

Appendix 1: Trial functions for the particular integral

RHS of differential equation	Trial function
$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ ($a_n \neq 0$, but a_i may be zero for $i \neq n$)	$b_n x^n + b_{n-1} x^{n-1} + \dots + b_1 x + b_0$ (if $a_i = 0$, we don't set $b_i = 0$)
$a e^{px}$	$b e^{px}$
$a_1 \sin px + a_2 \cos px$ (where one of the a_i may be zero)	$b_1 \sin px + b_2 \cos px$ (if $a_i = 0$, we don't set $b_i = 0$)

Notes

(i) If the CF is a multiple of the trial function, multiply by x .

(ii) For a combination of functions on the RHS, the trial function will be the corresponding combination of trial functions; eg for

$x^2 + \cos 3x - 2e^{2x}$ it would be

$$ax^2 + b + c \sin 3x + d \cos 3x + f e^{2x}$$

(and equating coefficients of $x^2, x, x^0, \sin 3x, \cos 3x$ & e^{2x})

Appendix 2: Auxiliary equation with repeated roots

Suppose that the homogeneous differential equation is

$$\frac{d^2 y}{dx^2} - 2k \frac{dy}{dx} + k^2 y = 0, \text{ so that the auxiliary equation is}$$

$$\lambda^2 - 2k\lambda + k^2 = 0 \text{ or } (\lambda - k)^2 = 0$$

Then consider the complementary function $y = (A + Bx)e^{kx}$.

$$\frac{dy}{dx} = B e^{kx} + k(A + Bx)e^{kx} = (B + kA + kBx)e^{kx}$$

and $\frac{d^2 y}{dx^2} = kB e^{kx} + k(B + kA + kBx)e^{kx}$ (differentiating the 1st expression for $\frac{dy}{dx}$)

$$= (2kB + k^2A + k^2Bx)e^{kx}$$

Then, substituting into the LHS of the differential equation gives

$$(2kB + k^2A + k^2Bx)e^{kx} - 2k(B + kA + kBx)e^{kx}$$

$$+ k^2(A + Bx)e^{kx}$$

$$= e^{kx}\{Bx(k^2 - 2k^2 + k^2) + A(k^2 - 2k^2 + k^2) + B(2k - 2k)\}$$

$$= 0, \text{ as required.}$$