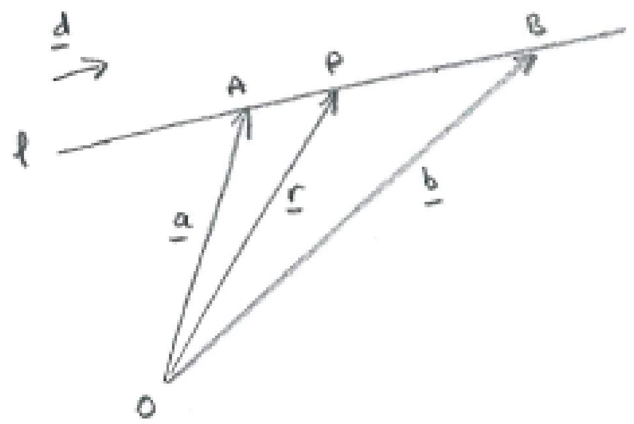
Vectors - Equation of line (6 pages; 18/9/20)

Contents

(1) Parametric form



The vector equation of the line l through the points A & B can be written in various (parametric) forms:

(a) $\underline{r} = \underline{a} + \lambda\underline{d}$

(b) $\underline{r} = \underline{a} + \lambda(\underline{b} - \underline{a})$

(c) $\underline{r} = (1-\lambda)\underline{a} + \lambda\underline{b}$

(a weighted average of $\underline{a}$ & $\underline{b}$; when $\lambda = 0, \underline{r} = \underline{a}$; when $\lambda = 1$,

$\underline{r} = \underline{b}$; when $\lambda = \frac{1}{2}$, $\underline{r}$ is the average of $\underline{a}$ & $\underline{b}$; the diagram shows $\lambda = \frac{1}{3}$)

(d) (in 2D case; similarly for 3D)

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ or $\begin{pmatrix} a_1 + \lambda d_1 \\ a_2 + \lambda d_2 \end{pmatrix}$

where $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\underline{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ is any vector in the direction from A to B

(normally d_1 & d_2 are chosen to be integers with no common factor)

Note the difference between (a) the vector equation of the line through the points A & B and (b) the vector $\overrightarrow{AB}$: The vector $\overrightarrow{AB}$ has magnitude |AB| (the distance between A & B) and is in the direction from A to B.

Whereas the vector equation of the line through A & B is the position vector $\underline{r}$ of a general point P on the line, with completely different magnitude and direction to that of the vector $\overrightarrow{AB}$.

[Note: The line from A to B, not extending beyond A and B is sometimes referred to as the 'line segment AB'.]

Exercise: If the line $\underline{r} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ can also be written as

$\underline{r} = \begin{pmatrix} 0 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 6 \end{pmatrix}$, find μ in terms of λ.

Solution

$2 + \lambda = -3\mu$ (1) & $3 - 2\lambda = 7 + 6\mu$ (2)

$(1) \Rightarrow \mu = -\frac{1}{3}(2 + \lambda)$

$[(2) \Rightarrow \mu = \frac{1}{6}(-4 - 2\lambda) = -\frac{1}{3}(2 + \lambda)$ also$]$

(2) Relation to the cartesian form

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \Rightarrow \lambda = \frac{x - a_1}{d_1} = \frac{y - a_2}{d_2}$

$\Rightarrow y = a_2 + \frac{d_2}{d_1} . (x - a_1)$

the straight line through (a_1, a_2) with gradient $\frac{d_2}{d_1}$

[In 3D: $\lambda = \frac{x - a_1}{d_1} = \frac{y - a_2}{d_2} = \frac{z - a_3}{d_3}$]

[Note: If the direction of $\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ is reversed, to give $\begin{pmatrix} -d_1 \\ -d_2 \end{pmatrix}$, then the gradient remains the same, as $\frac{-d_2}{-d_1} = \frac{d_2}{d_1}$]

Example: The line through the points (1, 0, 1) and (0, 1, 0)

$\underline{d} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

Hence $\underline{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ or $\begin{pmatrix} 1 - \lambda \\ \lambda \\ 1 - \lambda \end{pmatrix}$

and $\lambda = \frac{x - 1}{-1} = \frac{y - 0}{1} = \frac{z - 1}{-1}$

Special cases

Example 1: $\frac{x-2}{3} = \frac{y-4}{5}; z = 6$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$$

[Division by zero is undefined, so we cannot write

$\frac{x-2}{3} = \frac{y-4}{5} = \frac{z-6}{0}$]

The line is in the plane $z = 6$ (parallel to the line $\frac{x-2}{3} = \frac{y-4}{5}$ in the x-y plane).

Example 2: $\frac{x-2}{3} = \lambda; y = 1; z = 6$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The line is parallel to the x-axis, and passes through the point (2,1,6) (or any point of the form $(\mu, 1,6)$).

(3) Intersection of two lines

Example: To find the intersection of the lines

$$\underline{r} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 5 \\ -3 \end{pmatrix} \quad \text{and } \underline{r} = \begin{pmatrix} 7 \\ 9 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix},$$

solve $\begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix}$ for λ & μ.

If the lines don't meet, then a solution will not exist.

Here λ & μ turn out to be 2 & -1, giving $\underline{r} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} + 2\begin{pmatrix} -2 \\ 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ -5 \end{pmatrix}$

(or $\begin{pmatrix} 7 \\ 9 \\ -3 \end{pmatrix} - \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix}$),

so that the point of intersection is $(0, 8, -5)$.

(4) Direction Cosines

If the direction vector of a line is $\underline{d} = d_1\underline{i} + d_2\underline{j} + d_3\underline{k}$,

we can write $d_1 = |\underline{d}|cos\theta_1$, so that the **direction cosines** are

defined as $l_1(= cos\theta_1) = \frac{d_1}{|\underline{d}|}$, $l_2 = \frac{d_2}{|\underline{d}|}$ & $l_3 = \frac{d_3}{|\underline{d}|}$

and $\begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}$ is a unit vector

[Direction cosines are usually applied in the 3D case, where there isn't a gradient as such.]

Notes

(i) The letters l, m and n are often used instead of l_1, l_2 and l_3.

(ii) The **direction ratios** of a line are just d_1, d_2 and d_3 (or any 3 numbers in the same ratio).

(5) Vector product form

This only applies to 3D lines.

$\underline{r} = \underline{a} + \lambda\underline{d}$ can be written as $(\underline{r} - \underline{a}) \times \underline{d} = \underline{0}$

(since $\underline{r} - \underline{a}$ and $\underline{d}$ are parallel)

or $\underline{r} \times \underline{d} = \underline{a} \times \underline{d}$

eg line through (1, 0, 1) and (0, 1, 0):

$$\underline{d} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

$$\underline{a} \times \underline{d} = \begin{vmatrix} \underline{i} & 1 & -1 \\ \underline{j} & 0 & 1 \\ \underline{k} & 1 & -1 \end{vmatrix} = -i + k = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Thus equation is $\underline{r} \times \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Note: Textbooks sometimes write the determinant with the elements transposed (it gives the same result though).

To reconcile $\underline{r} \times \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ with $\underline{r} = \begin{pmatrix} 1-\lambda \\ \lambda \\ 1-\lambda \end{pmatrix}$:

LHS = $\begin{vmatrix} i & x & -1 \\ j & y & 1 \\ k & z & -1 \end{vmatrix} = (-y - z)\underline{i} - (-x + z)\underline{j} + (x + y)\underline{k}$

Hence $\begin{pmatrix} -y-z \\ x-z \\ x+y \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Let $y = \lambda$; then $x = 1 - \lambda$ and $z = 1 - \lambda$