Vectors Q3 (3/7/23)

Find the shortest distance between the lines

$\frac{x-1}{2} = \frac{y+3}{5} = \frac{z-2}{3}$ and $\frac{x}{1} = \frac{y-4}{2} = \frac{z+1}{2}$, identifying the points on the lines at which this shortest distance occurs.

Solution

Method 1

General points on the two lines are

$$\overrightarrow{OX} = \begin{pmatrix} 1+2\lambda \\ -3+5\lambda \\ 2+3\lambda \end{pmatrix} \text{ and } \overrightarrow{OY} = \begin{pmatrix} \mu \\ 4+2\mu \\ -1+2\mu \end{pmatrix}$$

At the closest approach of the two lines, $\overrightarrow{XY}$ will be perpendicular to both lines, so that

$$\overrightarrow{XY}.\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} = 0 \text{ and } \overrightarrow{XY}.\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 0,$$

so that $\begin{pmatrix} \mu-(1+2\lambda) \\ 4+2\mu-(-3+5\lambda) \\ -1+2\mu-(2+3\lambda) \end{pmatrix}.\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} = 0$ and

$$\begin{pmatrix} \mu-(1+2\lambda) \\ 4+2\mu-(-3+5\lambda) \\ -1+2\mu-(2+3\lambda) \end{pmatrix}.\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 0,$$

giving $(2\mu-2-4\lambda)+(35+10\mu-25\lambda)+(-9+6\mu-9\lambda)=0$
or $18\mu-38\lambda=-24$; ie $9\mu-19\lambda=-12$ (1)
and $(\mu-1-2\lambda)+(14+4\mu-10\lambda)+(-6+4\mu-6\lambda)=0$

$9\mu-18\lambda=-7$ (2)

Then (1) $-$ (2) $\Rightarrow -\lambda=-5$, so that $\lambda=5$ and, from (2),

$$\mu = \frac{1}{9}(18(5)-7) = \frac{83}{9}$$

So $\overrightarrow{OX} = \begin{pmatrix} 11 \\ 22 \\ 17 \end{pmatrix}$ and $\overrightarrow{OY} = \frac{1}{9}\begin{pmatrix} 83 \\ 202 \\ 157 \end{pmatrix}$

and $\overrightarrow{XY} = \frac{1}{9}\begin{pmatrix} 83-99 \\ 202-198 \\ 157-153 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} -16 \\ 4 \\ 4 \end{pmatrix} = \frac{4}{9}\begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}$,

so that $|\overrightarrow{XY}| = \frac{4}{9}\sqrt{16+1+1} = \frac{4\sqrt{18}}{9} = \frac{4\sqrt{2}}{3}$

Method 2

If $\underline{\hat{n}}$ is a unit vector perpendicular to both lines, then we need $\overrightarrow{OX}$ and $\overrightarrow{OY}$ such that $\overrightarrow{OX} + d\underline{\hat{n}} = \overrightarrow{OY}$, and the shortest distance will then be $|d|$.

A vector perpendicular to both lines is $\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{vmatrix} \underline{i} & 2 & 1 \\ \underline{j} & 5 & 2 \\ \underline{k} & 3 & 2 \end{vmatrix}$

$= \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$, so that $\underline{\hat{n}} = \frac{1}{\sqrt{18}}\begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$

Then $\overrightarrow{OX} + d\underline{\hat{n}} = \overrightarrow{OY}$ gives $\begin{pmatrix} 1+2\lambda \\ -3+5\lambda \\ 2+3\lambda \end{pmatrix} + D\begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} \mu \\ 4+2\mu \\ -1+2\mu \end{pmatrix}$,
where $D = \frac{d}{\sqrt{18}}$,

so that $2\lambda + 4D - \mu = -1$ (1)
$5\lambda - D - 2\mu = 7$ (2)
$3\lambda - D - 2\mu = -3$ (3)

Then (2) − (3) $\Rightarrow 2\lambda = 10$, so that $\lambda = 5$

and (1) & (2) become $4D - \mu = -11$ (4) and $-D - 2\mu = -18$ (5)

Then 2(4) − (5) $\Rightarrow 9D = -4$, so that $|d| = \sqrt{18}|D| = \frac{4\sqrt{18}}{9} = \frac{4\sqrt{2}}{3}$

and, from (1), $\mu = 10 - \frac{16}{9} + 1 = \frac{83}{9}$

and $\overrightarrow{OX}$ and $\overrightarrow{OY}$ can then be found, as in (i).

Method 3

Suppose that the two closest points are

$\overrightarrow{OX} = \begin{pmatrix} 1+2\lambda \\ -3+5\lambda \\ 2+3\lambda \end{pmatrix}$ and $\overrightarrow{OY} = \begin{pmatrix} \mu \\ 4+2\mu \\ -1+2\mu \end{pmatrix}$

A vector perpendicular to both lines is $\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{vmatrix} \underline{i} & 2 & 1 \\ \underline{j} & 5 & 2 \\ \underline{k} & 3 & 2 \end{vmatrix}$

$= \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$, and Y can be reached from X by travelling a certain distance in this direction.

Thus $\begin{pmatrix} 1+2\lambda \\ -3+5\lambda \\ 2+3\lambda \end{pmatrix} + k\begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} \mu \\ 4+2\mu \\ -1+2\mu \end{pmatrix}$,

or $\begin{pmatrix} 2\lambda + 4k - \mu \\ 5\lambda - k - 2\mu \\ 3\lambda - k - 2\mu \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ -3 \end{pmatrix}$

ie $\begin{pmatrix} 2 & 4 & -1 \\ 5 & -1 & -2 \\ 3 & -1 & -2 \end{pmatrix}\begin{pmatrix} \lambda \\ k \\ \mu \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ -3 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} \lambda \\ k \\ \mu \end{pmatrix} = \frac{1}{(0+45-27)}\begin{pmatrix} 0 & 4 & -2 \\ 9 & -1 & 14 \\ -9 & -1 & -22 \end{pmatrix}^T \begin{pmatrix} -1 \\ 7 \\ -3 \end{pmatrix}$

$$= \frac{1}{18}\begin{pmatrix} 0 & 9 & -9 \\ 4 & -1 & -1 \\ -2 & 14 & -22 \end{pmatrix}\begin{pmatrix} -1 \\ 7 \\ -3 \end{pmatrix} = \frac{1}{18}\begin{pmatrix} 90 \\ -8 \\ 166 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 45 \\ -4 \\ 83 \end{pmatrix}$$

Hence the two closest points are $\begin{pmatrix} 11 \\ 22 \\ 17 \end{pmatrix}$ and $\begin{pmatrix} \frac{83}{9} \\ \frac{202}{9} \\ \frac{157}{9} \end{pmatrix}$,

and the distance between them is $\left|-\frac{4}{9}\right|\sqrt{4^2 + (-1)^2 + (-1)^2}$

$$= \frac{4\sqrt{18}}{9} = \frac{4\sqrt{2}}{3}$$