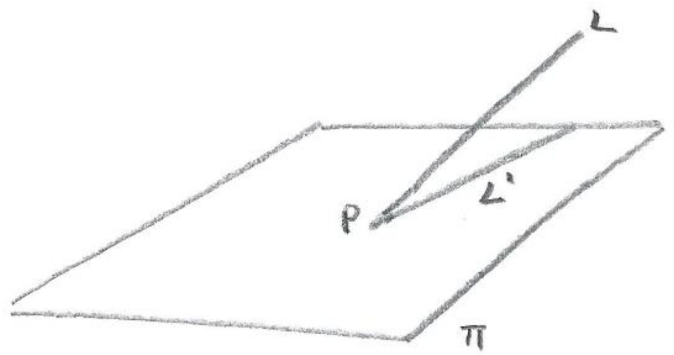
Vectors Q23 (3/7/23)

Given the plane Π: $3x + 2y - z = 6$ and the line

$L: \underline{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, let L' be the projection of L onto Π



(i) Find the point of intersection (P) of Π & L

(ii) Find the angle between Π & L

(iii) Find a vector that is parallel to Π and perpendicular to L

(iv) Find a vector equation for L'

(v) Find the angle between L and L'

Solution

(i) $3(1+2\lambda)+2(-\lambda)-(3+\lambda)=6$

$\Rightarrow 3\lambda = 6 \Rightarrow \lambda = 2$

So P is $\begin{pmatrix} 1+4 \\ -2 \\ 3+2 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix}$

(ii) The angle between Π & L is the angle between L and its projection onto the plane (ie the angle between L and L'), but is most easily determined by first finding the angle between L and the normal to the plane, and subtracting this from $\frac{\pi}{2}$

If θ is the angle between L and the normal to the plane, then

$$cos\theta = \frac{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}}{\sqrt{4+1+1}\sqrt{9+4+1}} = \frac{6-2-1}{\sqrt{6}\sqrt{14}} = \frac{3}{2\sqrt{21}} = \frac{3\sqrt{21}}{2(21)} = \frac{\sqrt{21}}{14}$$

The required angle is then $\arcsin\left(\frac{\sqrt{21}}{14}\right) = 19.107° = 19.1°$ (1dp)

(iii) [Note that "parallel to the plane" means parallel to a vector in the plane, and therefore perpendicular to the normal to the plane. The required vector is also perpendicular to the plane containing L and L'.]

As the required vector is perpendicular to both the normal to the plane and L, we can use the vector product:

$$\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{vmatrix} \underline{i} & 3 & 2 \\ \underline{j} & 2 & -1 \\ \underline{k} & -1 & 1 \end{vmatrix} = \underline{i} - 5\underline{j} - 7\underline{k}$$

[A useful check is that the scalar product with the original vectors is zero. Thus $3(1)+2(-5)+(-1)(-7)=0$]

(iv) L' passes through P and its direction is perpendicular to both

$\underline{i}-5\underline{j}-7\underline{k}$ (from (iii)) and the normal to the plane.

So its direction vector is

$$\begin{pmatrix}1\\-5\\-7\end{pmatrix}\times\begin{pmatrix}3\\2\\-1\end{pmatrix}=\begin{vmatrix}\underline{i} & 1 & 3\\ \underline{j} & -5 & 2\\ \underline{k} & -7 & -1\end{vmatrix}=19\underline{i}-20\underline{j}+17\underline{k}$$

So a vector equation of L' is: $\underline{r}=\begin{pmatrix}5\\-2\\5\end{pmatrix}+\lambda\begin{pmatrix}19\\-20\\17\end{pmatrix}$, from (i).

(v) If the required angle is ϕ, then

$$cos\phi=\frac{\begin{pmatrix}2\\-1\\1\end{pmatrix}.\begin{pmatrix}19\\-20\\17\end{pmatrix}}{\sqrt{4+1+1}\sqrt{361+400+289}}=\frac{38+20+17}{\sqrt{6}\sqrt{1050}}$$

$$=\frac{75}{2\sqrt{3}\sqrt{525}}=\frac{75}{2\sqrt{3}(5)\sqrt{21}}=\frac{15}{2(3)\sqrt{7}}=\frac{5\sqrt{7}}{14}$$

and hence $\phi=\arccos\left(\frac{5\sqrt{7}}{14}\right)=19.107°=19.1°$ (1dp) (as in (ii))