Vectors Q19 (3/7/23)

(i) Show the lines $\frac{x-1}{2} = \frac{y+3}{5} = \frac{z-2}{3}$ and $\frac{x}{1} = \frac{y-4}{2} = \frac{z+1}{2}$ are skew.

(ii) Find the shortest distance between the lines and identify the points on the lines at which this shortest distance occurs.

Solution

(i) The lines can be rewritten in parametric form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+2\lambda \\ -3+5\lambda \\ 2+3\lambda \end{pmatrix} \text{ and } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \mu \\ 4+2\mu \\ -1+2\mu \end{pmatrix}$$

A point of intersection would then satisfy

$1 + 2\lambda = \mu \quad (1)$
$-3 + 5\lambda = 4 + 2\mu \ (2)$
$2 + 3\lambda = -1 + 2\mu \quad (3)$

Substituting from (1) into (2) & (3) gives:

$-3 + 5\lambda = 4 + 2(1 + 2\lambda)$ or $-9 = -\lambda$, so that $\lambda = 9$

and $2 + 3\lambda = -1 + 2(1 + 2\lambda)$ or $1 = \lambda$,

and so there is no point of intersection.

Also, the direction vectors of the lines are not parallel, and so the lines are skew.

(ii) [There are various methods for finding the shortest distance, but not all of them find the points on the lines where the shortest distance occurs. The first method given below is relatively straightforward, and doesn't involve the vector product.]

Method 1

From (i), general points on the two lines are

$$\overrightarrow{OX} = \begin{pmatrix} 1+2\lambda \\ -3+5\lambda \\ 2+3\lambda \end{pmatrix} \text{ and } \overrightarrow{OY} = \begin{pmatrix} \mu \\ 4+2\mu \\ -1+2\mu \end{pmatrix}$$

At the closest approach of the two lines, $\overrightarrow{XY}$ will be perpendicular to both lines, so that

$\overrightarrow{XY}.\begin{pmatrix}2\\5\\3\end{pmatrix} = 0$ and $\overrightarrow{XY}.\begin{pmatrix}1\\2\\2\end{pmatrix} = 0$, so that

$$\begin{pmatrix}\mu-(1+2\lambda)\\4+2\mu-(-3+5\lambda)\\-1+2\mu-(2+3\lambda)\end{pmatrix}.\begin{pmatrix}2\\5\\3\end{pmatrix} = 0 \text{ and}$$
$$\begin{pmatrix}\mu-(1+2\lambda)\\4+2\mu-(-3+5\lambda)\\-1+2\mu-(2+3\lambda)\end{pmatrix}.\begin{pmatrix}1\\2\\2\end{pmatrix} = 0,$$

giving $(2\mu-2-4\lambda)+(35+10\mu-25\lambda)+(-9+6\mu-9\lambda) = 0$ or $18\mu-38\lambda = -24$; ie $9\mu-19\lambda = -12$ (1)

and $(\mu-1-2\lambda)+(14+4\mu-10\lambda)+(-6+4\mu-6\lambda) = 0$

$9\mu-18\lambda = -7$ (2)

Then (1) − (2) $\Rightarrow -\lambda = -5$, so that $\lambda = 5$ and, from (2),

$\mu = \frac{1}{9}(18(5)-7) = \frac{83}{9}$

So $\overrightarrow{OX} = \begin{pmatrix}11\\22\\17\end{pmatrix}$ and $\overrightarrow{OY} = \frac{1}{9}\begin{pmatrix}83\\202\\157\end{pmatrix}$

and $\overrightarrow{XY} = \frac{1}{9}\begin{pmatrix}83-99\\202-198\\157-153\end{pmatrix} = \frac{1}{9}\begin{pmatrix}-16\\4\\4\end{pmatrix} = \frac{4}{9}\begin{pmatrix}-4\\1\\1\end{pmatrix}$,

so that $\left|\overrightarrow{XY}\right| = \frac{4}{9}\sqrt{16+1+1} = \frac{4\sqrt{18}}{9} = \frac{4\sqrt{2}}{3}$

Method 2 (using the vector product)

If $\underline{\hat{n}}$ is a unit vector perpendicular to both lines, then we need $\overrightarrow{OX}$ and $\overrightarrow{OY}$ such that $\overrightarrow{OX} + d\underline{\hat{n}} = \overrightarrow{OY}$, and the shortest distance will then be $|d|$.

A vector perpendicular to both lines is $\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{vmatrix} \underline{i} & 2 & 1 \\ \underline{j} & 5 & 2 \\ \underline{k} & 3 & 2 \end{vmatrix}$

$= \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$, so that $\underline{\hat{n}} = \frac{1}{\sqrt{18}} \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$

Then $\overrightarrow{OX} + d\underline{\hat{n}} = \overrightarrow{OY}$ gives $\begin{pmatrix} 1 + 2\lambda \\ -3 + 5\lambda \\ 2 + 3\lambda \end{pmatrix} + D \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} \mu \\ 4 + 2\mu \\ -1 + 2\mu \end{pmatrix}$,

where $D = \frac{d}{\sqrt{18}}$,

so that $2\lambda + 4D - \mu = -1$ (1)
$5\lambda - D - 2\mu = 7$ (2)
$3\lambda - D - 2\mu = -3$ (3)

Then $(2) - (3) \Rightarrow 2\lambda = 10$, so that $\lambda = 5$

and (1) & (2) become $4D - \mu = -11$ (4) and $-D - 2\mu = -18$ (5)

Then $2(4) - (5) \Rightarrow 9D = -4$, so that $|d| = \sqrt{18}|D| = \frac{4\sqrt{18}}{9} = \frac{4\sqrt{2}}{3}$

and, from (1), $\mu = 10 - \frac{16}{9} + 1 = \frac{83}{9}$

and $\overrightarrow{OX}$ and $\overrightarrow{OY}$ can then be found, as in (i).