Vectors Q17 (3/7/23)

Find the line that is the reflection of the line $\frac{x-2}{3} = \frac{y}{4} = \frac{z+1}{1}$ in the plane $x - 2y + z = 4$

Solution

Let the intersection of the line and the plane be P, and suppose that Q is some other point on the line. Then we can find the reflection of Q in the plane (Q' say), by dropping a perpendicular from Q onto the plane, and then the required line will pass through P and Q'.

Writing the equation of the line as $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ and substituting into the equation of the plane:

$$(2 + 3\lambda) - 2(4\lambda) + (-1 + \lambda) = 4 \Rightarrow -4\lambda = 3;\ \lambda = -\tfrac{3}{4}$$

so that P is $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} - \frac{3}{4}\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ -3 \\ -\frac{7}{4} \end{pmatrix}$

Setting $\lambda = 1$ (say), we can take Q to be $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}$

Now consider the perpendicular line dropped from Q onto the plane. Its direction vector is that of the normal to the plane, and so it has equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Let R be the point where the perpendicular line intersects the plane. Substituting into the equation of the plane gives:

$$(5 + \lambda) - 2(4 - 2\lambda) + (\lambda) = 4 \Rightarrow 6\lambda = 7;\ \lambda = \tfrac{7}{6}$$

So R is $\begin{pmatrix}5\\4\\0\end{pmatrix}+\frac{7}{6}\begin{pmatrix}1\\-2\\1\end{pmatrix}$, and Q' will be $\begin{pmatrix}5\\4\\0\end{pmatrix}+2\left(\frac{7}{6}\right)\begin{pmatrix}1\\-2\\1\end{pmatrix}=\begin{pmatrix}\frac{22}{3}\\-\frac{2}{3}\\\frac{7}{3}\end{pmatrix}$

Then , as P is $\begin{pmatrix}-\frac{1}{4}\\-3\\-\frac{7}{4}\end{pmatrix}$, the equation of the reflected line will be:

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-\frac{1}{4}\\-3\\-\frac{7}{4}\end{pmatrix}+\lambda\left[\begin{pmatrix}\frac{22}{3}\\-\frac{2}{3}\\\frac{7}{3}\end{pmatrix}-\begin{pmatrix}-\frac{1}{4}\\-3\\-\frac{7}{4}\end{pmatrix}\right]=\frac{1}{12}\begin{pmatrix}-3+\lambda(88+3)\\-36+\lambda(-8+36)\\-21+\lambda(28+21)\end{pmatrix}$$

$$\frac{1}{12}\begin{pmatrix}-3+91\lambda\\-36+28\lambda\\-21+49\lambda\end{pmatrix}$$

or, in cartesian form: $\frac{x+\frac{3}{12}}{91}=\frac{y+\frac{36}{12}}{28}=\frac{z+\frac{21}{12}}{49}$ or $\frac{x+\frac{3}{12}}{13}=\frac{y+\frac{36}{12}}{4}=\frac{z+\frac{21}{12}}{7}$