Vectors Q14 (3/7/23)

Find the volume of the tetrahedron with corners
$(2, 1, 3), (-1, 5, 0), (4, 4, 7), (8, 2, 2)$

Solution

Method 1

Label the corners as follows:

$A(2,1,3), B(-1,5,0), C(4,4,7), D(8,2,2)$

Then volume $= \frac{1}{3} \cdot \frac{1}{2} |\overrightarrow{AB} . (\overrightarrow{AC} \times \overrightarrow{AD})|$

(based on $\frac{1}{3}$ × area of triangle ABC × perpendicular height)

and $\overrightarrow{AB} . (\overrightarrow{AC} \times \overrightarrow{AD}) = \begin{vmatrix} -3 & 2 & 6 \\ 4 & 3 & 1 \\ -3 & 4 & -1 \end{vmatrix}$

$= -3(-7) - 4(-26) - 3(-16) = 21 + 104 + 48 = 173$

So volume is $\frac{173}{6}$ units^3.

Method 2a (much longer, but good practice!)

Volume $= \frac{1}{3}$ × area of base ABC × perpendicular height

Area of base ABC $= \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|$

and $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & -3 & 2 \\ \underline{j} & 4 & 3 \\ \underline{k} & -3 & 4 \end{vmatrix} = \begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix}$,

so that Area of base ABC $= \frac{1}{2}\sqrt{25^2 + 6^2 + (-17)^2} = \frac{5}{2}\sqrt{38}$

The perpendicular height is the shortest distance from D to the plane ABC.

A normal to the plane ABC is $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix}$ (already calculated).

And the equation of the plane ABC is

$$\underline{r}.\begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.\begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix} = 50 + 6 - 51 = 5\,,$$

taking $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ as a point in the plane.

Let the point of intersection of the perpendicular from D onto the plane ABC be P, given by the following point on the perpendicular:

$$\begin{pmatrix} 8 \\ 2 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix}$$

As P lies in the plane ABC,

$$\left(\begin{pmatrix} 8 \\ 2 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix}\right).\begin{pmatrix} 25 \\ 6 \\ -17 \end{pmatrix} = 5$$

Then $178 + 950\lambda = 5$, so that $\lambda = -\frac{173}{950}$

and the perpendicular height is $|\lambda|\left|\begin{matrix} 25 \\ 6 \\ -17 \end{matrix}\right|$

$$= \frac{173}{950}\sqrt{25^2 + 6^2 + (-17)^2} = \frac{173}{950}\,.5\sqrt{38} = \frac{173}{190}\sqrt{38}$$

Hence the volume of the tetrahedron is

$$\frac{1}{3}.\frac{5}{2}\sqrt{38}\,.\frac{173}{190}\sqrt{38} = \frac{173}{6}\ \text{units}^3$$

Method 2b (even longer)

As Method 2a, but determining λ as follows:

For P to be a point in the plane ABC,

$$\begin{pmatrix}8\\2\\2\end{pmatrix}+\lambda\begin{pmatrix}25\\6\\-17\end{pmatrix}=\begin{pmatrix}2\\1\\3\end{pmatrix}+\mu\begin{pmatrix}-3\\4\\-3\end{pmatrix}+\theta\begin{pmatrix}2\\3\\4\end{pmatrix},$$

as $\overrightarrow{OA}=\begin{pmatrix}2\\1\\3\end{pmatrix}$ is a point in the plane, and $\overrightarrow{AB}=\begin{pmatrix}-3\\4\\-3\end{pmatrix}$ and

$\overrightarrow{AC}=\begin{pmatrix}2\\3\\4\end{pmatrix}$ are directions parallel to the plane

Then $\begin{pmatrix}25&3&-2\\6&-4&-3\\-17&3&-4\end{pmatrix}\begin{pmatrix}\lambda\\\mu\\\theta\end{pmatrix}=\begin{pmatrix}-6\\-1\\1\end{pmatrix}$

$$\begin{vmatrix}25&3&-2\\6&-4&-3\\-17&3&-4\end{vmatrix}=25(25)-6(-6)-17(-17)=950$$

$$\begin{pmatrix}25&3&-2\\6&-4&-3\\-17&3&-4\end{pmatrix}^{-1}=\frac{1}{950}\begin{pmatrix}25&75&-50\\6&-134&-126\\-17&63&-118\end{pmatrix}^{T}$$

So $\begin{pmatrix}\lambda\\\mu\\\theta\end{pmatrix}=\frac{1}{950}\begin{pmatrix}25&6&-17\\75&-134&63\\-50&-126&-118\end{pmatrix}\begin{pmatrix}-6\\-1\\1\end{pmatrix}$

$$= \frac{1}{950}\begin{pmatrix} -173 \\ -253 \\ 308 \end{pmatrix},$$

so that $\lambda = -\frac{173}{950}$