Vectors Q10 (3/7/23)

(i)(a) Find the acute angle between the line $\frac{x}{2} = \frac{y+1}{-3} = \frac{z-2}{1}$ and the plane $x + y - 2z = 5$

(b) Show that the same angle is obtained if the line is written in the form

$\frac{x}{-2} = \frac{y+1}{3} = \frac{z-2}{-1}$ (ie without rearranging into the form in (a))

(ii)(a) Find the acute angle between the planes $x + 4y - 3z = 7$

and $x - y + 4z = 2$

(b) Find the acute angle between the planes $x + 4y - 3z = 7$ and $-x + y - 4z = 2$ (again, without rearranging the equation)

Solution

(i)(a) The angle between the line and the normal to the plane is given by

$$\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} . \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \sqrt{14}\sqrt{6}\, cos\theta, \text{ so that } cos\theta = \frac{-3}{\sqrt{14}\sqrt{6}} = -0.32733$$

and $\theta = 109.107°$

The acute angle between these vectors is then $180 - 109.107 = 70.893°$

The acute angle between the line and plane is then

$90 - 70.893 = 19.1°$ (1dp)

(b) $\begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} . \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \sqrt{14}\sqrt{6}\, cos\theta \Rightarrow cos\theta = \frac{3}{\sqrt{14}\sqrt{6}} = 0.32733$

and $\theta = 70.893°$

As we have already found the acute angle between the line and the normal, the acute angle between the line and the plane is $90 - 70.893 = 19.1°$ (1dp)

(ii) The angle between the normals to the planes is given by

$$\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} = \sqrt{26}\sqrt{18}\, cos\theta, \text{ so that } cos\theta = \frac{-15}{\sqrt{26}\sqrt{18}} = -0.69338$$

and $\theta = 133.898°$

The acute angle between the planes themselves is $180 - 133.898 = 46.1°$

(ii)(b) The angle between the normals to the planes is given by

$$\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} . \begin{pmatrix} -1 \\ 1 \\ -4 \end{pmatrix} = \sqrt{26}\sqrt{18}\, cos\theta, \text{ so that } cos\theta = \frac{15}{\sqrt{26}\sqrt{18}} = 0.69338$$

and $\theta = 46.1°$

The acute angle between the planes is also 46.1°.