Vectors - Angles (5 pages; 4/8/18)

(1) Angle between a line and a plane

To find the acute angle between the line with equation

$\underline{r} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ and the plane with equation $\underline{r}.\begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix} = 1$

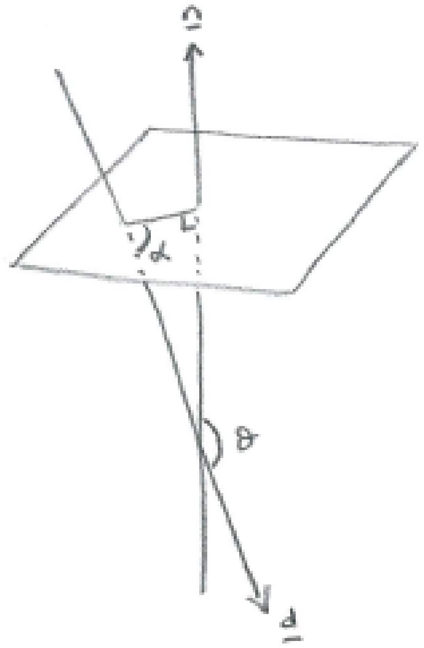
The two direction vectors in this case are $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix}$

Then $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}.\begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix} = \left|\begin{matrix} 1 \\ 2 \\ -1 \end{matrix}\right|\left|\begin{matrix} 5 \\ 1 \\ -1 \end{matrix}\right| cos\theta$ (*),

so that $cos\theta = \frac{(-1)(5)+(-2)(1)+(1)(-1)}{\sqrt{1+4+1}\sqrt{25+1+1}} = \frac{-8}{\sqrt{162}} = -0.62854$

This gives $\theta = 128.9°$

Whether θ is acute or obtuse depends on the relative directions of the normal vector to the plane ($\underline{n}$) and the direction vector of the line ($\underline{d}$) - see the diagram below.



The angle we want (between the plane and the line) is α in the diagram.

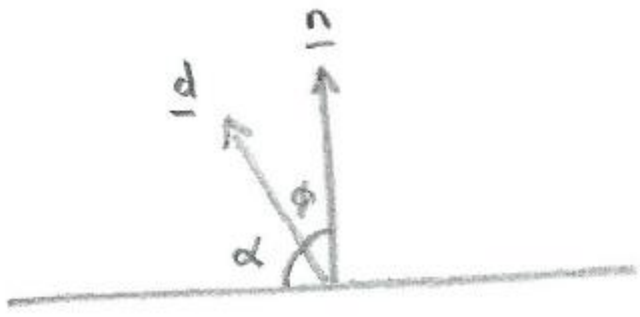
Thus $\alpha = 90 - (180 - 128.9) = 38.9°$

If the directions of either $\underline{n}$ or $\underline{d}$ are reversed, then the calculated angle θ between $\underline{n}$ and $\underline{d}$ becomes the acute angle $180 - 128.9 = 51.1°$, and $\alpha = 90 - 51.1 = 38.9°$

(If both of the directions of $\underline{n}$ or $\underline{d}$ are reversed, then $\theta = 128.9°$ again.)

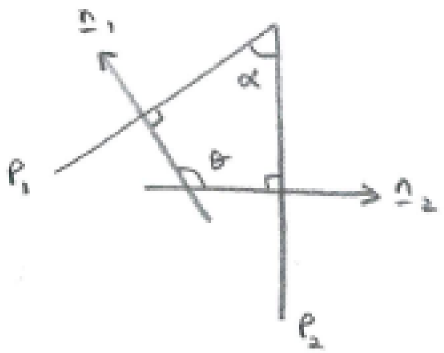
So, if the scalar product angle θ is acute, $\alpha = 90 - \theta$, whilst if θ is obtuse, $\alpha = 90 - (180 - \theta)$.

In other words, find the acute angle between $\underline{n}$ and $\underline{d}$, and subtract it from 90 (as in the simple diagram below).



(2) Angle between two planes

If we need to find the angle between two planes, then the angle in question will be α in the diagram below. (It will usually be the acute angle between the planes that is required.)



For the configuration of the normals shown in the diagram, the angle obtained from their scalar product (θ) is obtuse. In this case, $\alpha = 180 - \theta$. If either of the normals is reversed in direction, then the angle obtained from the scalar product is acute, and θ in the diagram is obtained by subtracting the scalar product angle from 180. In this case, α equals the scalar product angle. (If both of the normals are reversed in direction, then θ in the diagram equals the scalar product angle.)

So, if the scalar product angle is acute, α equals the scalar product angle. If the scalar product angle is obtuse, then α equals 180 minus the scalar product angle.

Or: α just equals the acute angle between the two normals.

(3) Vector perpendicular to a given (2D) vector

Example

Given direction vector $\begin{pmatrix}2\\3\end{pmatrix}$: gradient is $\frac{3}{2}$; hence perpendicular gradient $= -\frac{2}{3}$ and perpendicular direction vector is $\begin{pmatrix}-3\\2\end{pmatrix}$ or $\begin{pmatrix}3\\-2\end{pmatrix}$

(4) Vector perpendicular to two given (3D) vectors

Let given vectors be $\underline{a}$ and $\underline{b}$

Method 1

$\underline{a} \times \underline{b}$

Method 2

Let $\underline{d} = \begin{pmatrix}d_1\\d_2\\d_3\end{pmatrix}$ be the required vector.

Then eliminate two of d_1, d_2 & d_3 from $\underline{d}.\underline{a} = 0$ and $\underline{d}.\underline{b} = 0$ (*)

to give a direction vector in terms of parameter d_1, d_2 *or* d_3.

eg $\begin{pmatrix} d_1 \\ 2d_1 \\ 3d_1 \end{pmatrix}$, which is equivalent to the direction vector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

(Note: the form of eq'ns (*) ensures that d_2 and d_3 will be multiples of d_1.)