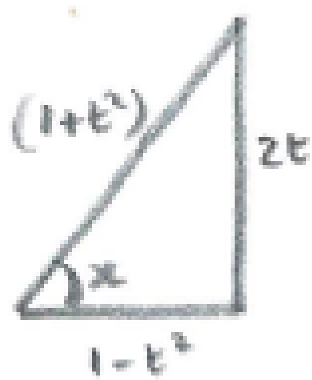
Trigonometry - t formulae (4 pages; 24/3/20)

[See also "Integration Methods" & "Trig. Exercises - Part 2".]

(1) The $t = tan(\frac{x}{2})$ substitution can be useful (albeit mainly as a method of last resort) for awkward integrals - enabling a complicated combination of trig. ratios to be converted to a combination of powers of t. Equations can also sometimes be solved by this method.

(2) $t = tan\left(\frac{x}{2}\right) \Rightarrow \tan x = \frac{2t}{1-t^2}$ (by the double angle formula)

Other trig. ratios can be read off the right-angled triangle in the diagram:



(Note that $(2t)^2 + (1-t^2)^2 = 4t^2 + 1 - 2t^2 + t^4$
$= 1 + 2t^2 + t^4 = (1+t^2)^2$)

Thus, $sinx = \frac{2t}{1+t^2}$ and $cosx = \frac{1-t^2}{1+t^2}$

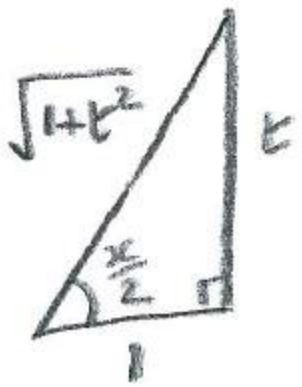
(3) The above derivation (using Pythagoras) is only valid for $x < \frac{\pi}{2}$. For larger angles, the same formulae hold, but an algebraic approach is necessary:

$sinx = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = \frac{2\tan(\frac{x}{2})}{sec^2(\frac{x}{2})} = \frac{2t}{1+t^2}$

and $cosx = cos^2\left(\frac{x}{2}\right) - sin^2\left(\frac{x}{2}\right) = \frac{1-tan^2(\frac{x}{2})}{sec^2(\frac{x}{2})} = \frac{1-t^2}{1+t^2}$

(4) Note that $t = tan\left(\frac{x}{2}\right)$ isn't defined when x is an odd multiple of π. When the substitution is used to solve equations, these values of x will have to be considered separately.

(5) Be careful not to confuse the triangle in (1) with the following triangle, which can be used to find expressions in terms of t for $sin(\frac{x}{2})$ and $cos(\frac{x}{2})$



Thus, $\sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1+t^2}}$ and $\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+t^2}}$

Alternative derivation of $\sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1+t^2}}$:

Let $\theta = \frac{x}{2}$, so that $tan\theta = t$

Then $\sin\left(\frac{x}{2}\right) = sin\theta = \frac{tan\theta}{sec\theta} = \frac{t}{\sqrt{1+tan^2\theta}} = \frac{t}{\sqrt{1+t^2}}$

(6) **Exercise**: Use the $t = tan(\frac{x}{2})$ substitution to find θ (in terms of a, b & c) when $acos\theta + bsin\theta = c$ $(0 \leq \theta < \pi)$ and $a^2 + b^2 = c^2$ $(a \neq 0)$

[Note: This eq'n is more easily solved by writing the LHS as $Rsin(\theta + \alpha)$.]

Solution

$$acos\theta + bsin\theta = c \Rightarrow a\left(\frac{1-t^2}{1+t^2}\right) + b\left(\frac{2t}{1+t^2}\right) = c$$

$$\Rightarrow a(1 - t^2) + 2bt = c(1 + t^2)$$

$$\Rightarrow t^2(c + a) - 2bt + c - a = 0$$

$$\Rightarrow t = \frac{2b \pm \sqrt{4b^2 - 4(c+a)(c-a)}}{2(c+a)} = \frac{b \pm \sqrt{b^2 - (c^2 - a^2)}}{c+a} = \frac{b}{a+c}$$

$$\text{Then } tan\theta = \frac{2t}{1-t^2} = \frac{(\frac{2b}{a+c})}{1-\left(\frac{b}{a+c}\right)^2} = \frac{2b(a+c)}{(a+c)^2 - b^2}$$

$$= \frac{2b(a+c)}{2a^2+2ac} = \frac{b}{a}$$

Thus, $\theta = tan^{-1}(\frac{b}{a})$

(7) If $t = tan(\frac{x}{2})$ and $T = tan(\frac{\frac{\pi}{2}-x}{2})$, find t in terms of T, and T in terms of t.

Solution

$$T = tan\left(\frac{\frac{\pi}{2}-x}{2}\right) = \tan\left(\frac{\pi}{4} - \frac{x}{2}\right) = \frac{\tan\left(\frac{\pi}{4}\right) - \tan(\frac{x}{2})}{1+\tan\left(\frac{\pi}{4}\right)\tan(\frac{x}{2})} = \frac{1-t}{1+t}$$

And hence $T(1 + t) = 1 - t$, so that $t(T + 1) = 1 - T$

and $t = \frac{1-T}{1+T}$ (or by symmetry, considering the right-angled triangle with other angles x and $\frac{\pi}{2} - x$).