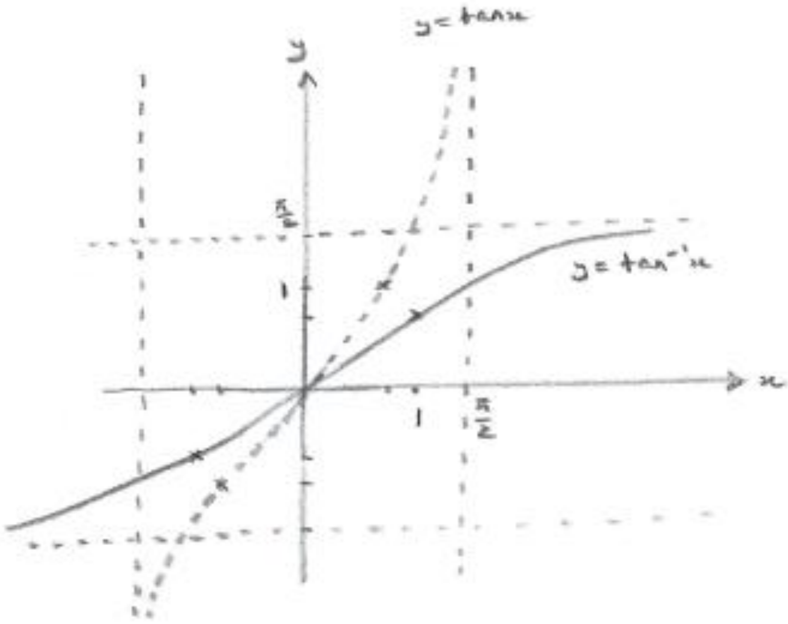
Trigonometry – Inverse Functions (5 pages; 15/4/21)

(1) $y = tan^{-1}x$ $(or\ arctanx)$



(i) The scales have to be in radians in order for these graphs to be reflections of each other in $y = x$.

(ii) In order to establish the gradient of $y = tanx$ at the Origin:

$\frac{d}{dx}(\tan x) = sec^2x$ = 1 when $x = 0$

(this assumes that the angle is in radians)

[The above graph hasn't been drawn that well: the gradients of both $y = tanx$ and $y = tan^{-1}x$ are intended to be 1 at the Origin.]

(iii) In order for $y = tan^{-1}x$ to be a 1-1 mapping (and therefore a function), the domain of $y = tanx$ has to be limited to $(-\frac{\pi}{2}, \frac{\pi}{2})$.

(iv) $\frac{d}{dx}\ tan^{-1}x$

Let $y = tan^{-1}x$

Then $tany = x$ and $sec^2y\ \frac{dy}{dx} = 1$ (differentiating implicitly wrt x) [alternatively, differentiate wrt y, to give $sec^2y\ = \frac{dx}{dy}$ and take the reciprocal]

And as $sec^2y\ = tan^2y + 1 = x^2 + 1,\ \frac{dy}{dx} = \frac{1}{1+x^2}$

Also $\int \frac{1}{1+x^2}\ dx = tan^{-1}x + c$

[See also "Integration methods".]

(v) Features of $\frac{dy}{dx}$ (in agreement with graph):

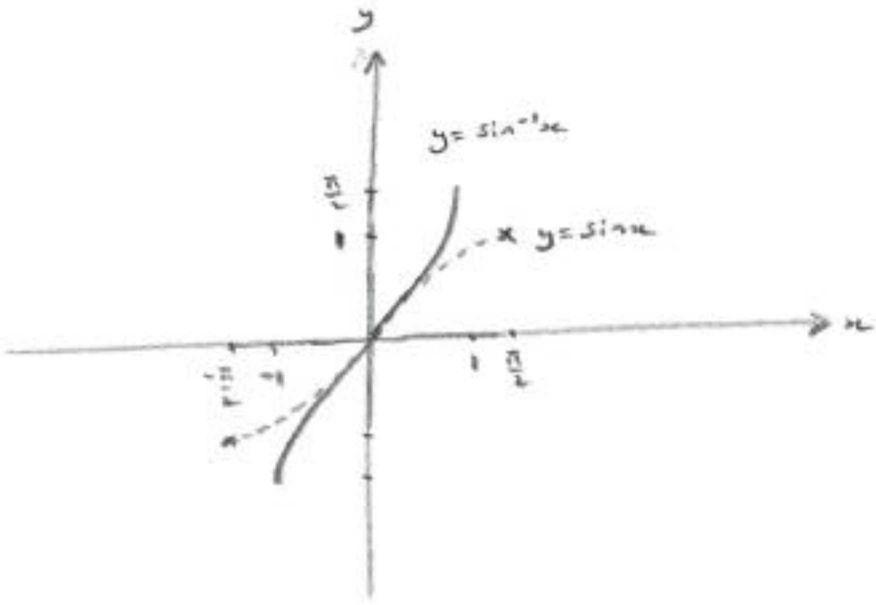
(a) $\frac{dy}{dx} \to 0$ as $x \to \pm\infty$

(b) $\frac{dy}{dx}$ is always positive

(c) $\frac{dy}{dx}$= 1 when $x = 0$

(vi) $tanx = y\ \to x = arctany + n\pi$

(2) $y = sin^{-1}x$ $(or\ arcsinx)$



(i) In order for $y = sin^{-1}x$ to be a 1-1 mapping (and therefore a function), the domain of $y = sinx$ has to be limited to $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

(ii) $\frac{d}{dx}(\sin x) = 1\ when\ x = 0$

(iii) $sinx = y\ \rightarrow x = (arcsiny\ or\ \pi - arcsiny) + 2n\pi$

or $x = n\pi + (-1)^n arcsiny$

(iv) $\frac{d}{dx}\ sin^{-1}x$

Let $y = sin^{-1}x$, so that $siny = x$ and $cosy\frac{dy}{dx} = 1$

Hence $\frac{dy}{dx} = \frac{1}{cosy} = \frac{1}{\sqrt{1-sin^2y}} = \frac{1}{\sqrt{1-x^2}}$, taking the positive root, as y is restricted to the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$, where $cosy > 0$ (assuming that $\frac{dy}{dx}$ is defined, so that $cosy \neq 0$)

(also $\frac{dy}{dx} > 0$ from the graph).

Also $\int \frac{1}{\sqrt{1-x^2}}\, dx = sin^{-1}x$ (see "Integration Methods").

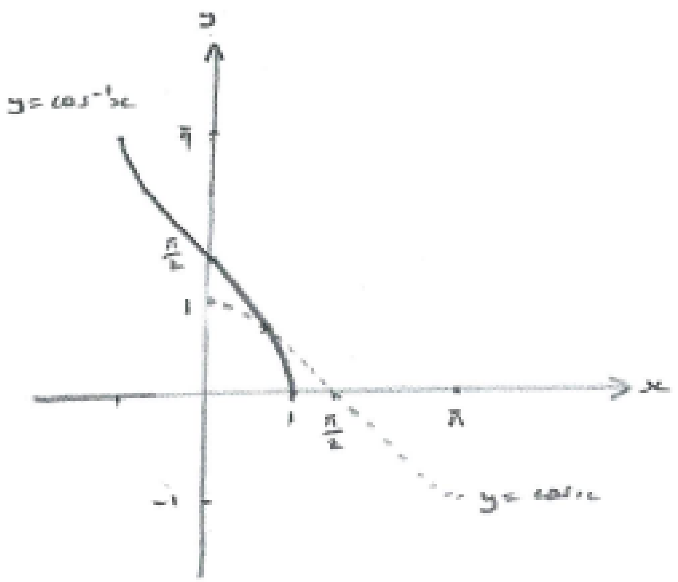
(v) Features of $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

(a) $\frac{dy}{dx} \to \infty$ as $x \to \pm 1$

(b) $\frac{dy}{dx}$ is always positive

(c) $\frac{dy}{dx}$ is undefined when $x \leq -1$ or $x \geq 1$

(3) $y = cos^{-1}x$ $(or\ arccosx)$



(i) In order for $y = cos^{-1}x$ to be a 1-1 mapping (and therefore a function), the domain of $y = cosx$ has to be limited to $[0, \pi]$.

(ii) $cosx = y \rightarrow x = (arccosy \;\; or \; 2\pi - arccosy) + 2n\pi$

or $\pm arccosy + 2n\pi$

(iii) $\frac{d}{dx} \; cos^{-1}x$

Let $y = cos^{-1}x$, so that $cosy = x$ and $-siny\frac{dy}{dx} = 1$

Hence $\frac{dy}{dx} = \frac{-1}{siny} = \frac{-1}{\sqrt{1-cos^2y}} = \frac{-1}{\sqrt{1-x^2}}$

taking the positive root , as y is restricted to the range $[0 \, , \, \pi]$,

when $siny > 0$ (assuming that $\frac{dy}{dx}$ is defined, so that $siny \neq 0$)

(also $\frac{dy}{dx} < 0$ from the graph)

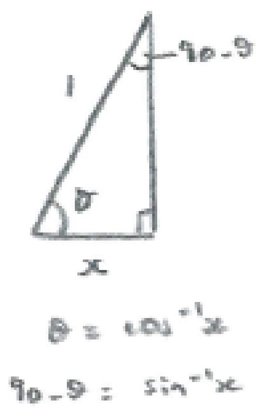
Also $\int \frac{1}{\sqrt{1-x^2}} \, dx = -cos^{-1}x$ (see "Integration Methods").

Note: $cos^{-1}x + sin^{-1}x = \frac{\pi}{2}$ (see diagram),

so that $-cos^{-1}x + c = sin^{-1}x - \frac{\pi}{2} + c$

Thus the two alternative expressions for $\int \frac{1}{\sqrt{1-x^2}} \, dx$

($sin^{-1}x \; \& - cos^{-1}x$) differ by a constant.



(iv) $y = cos^{-1}x$ is the reflection of $y = sin^{-1}x$ in $y = \frac{\pi}{4}$, *since*

$cos^{-1}x + sin^{-1}x = \frac{\pi}{2}$