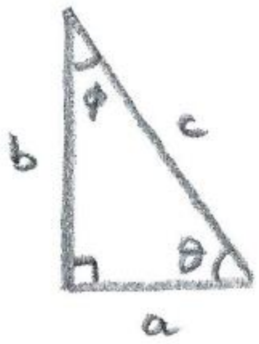
Trigonometry - Important Ideas (STEP) (5 pages; 15/4/21)

(1) Relation between sin and cos



Referring to the diagram,

$sin\theta = \frac{b}{c} = cos\phi = cos\ (90° - \theta)$

and $cos\theta = \frac{a}{c} = sin\phi = sin(90° - \theta)$

(The 'co' in cosine stands for 'complementary', because θ and $90° - \theta$ are described as complementary angles.)

(2) Key Results

(A) Compound Angle formulae

$sin(\theta + \phi) = sin\theta cos\phi + cos\theta sin\phi$

$cos(\theta + \phi) = cos\theta cos\phi - sin\theta sin\phi$

(B) $sin(\theta \pm 360°) = sin\theta$; $cos(\theta \pm 360°) = cos\theta$

$cos(-\theta) = cos\theta$; $sin(-\theta) = -sin\theta$

$sin(180° - \theta) = sin\theta$; $cos(180° - \theta) = -cos\theta$

$sin\theta = \cos(90° - \theta)$; $cos\theta = sin(90° - \theta)$

(C) Translations

$sin(\theta + 90°)$ is $sin\theta$ translated 90° to the left, which is $cos\theta$

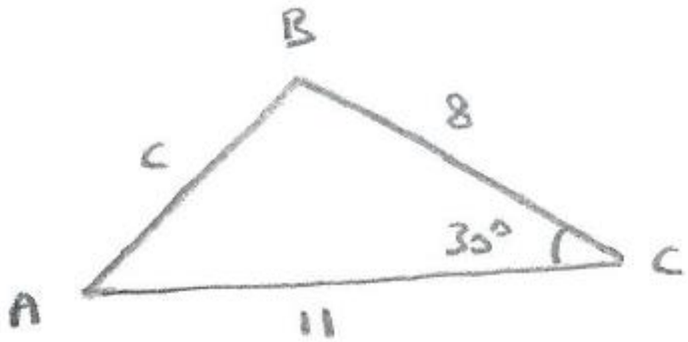
$sin(\theta - 90°)$ is $sin\theta$ translated 90° to the right, which is $-cos\theta$

$cos(\theta + 90°)$ is $cos\theta$ translated 90° to the left, which is $-sin\theta$

$cos(\theta - 90°)$ is $cos\theta$ translated 90° to the right, which is $sin\theta$

(3) As $sin\theta = sin\ (180° - \theta)$, we have to be careful when using the Sine rule to determine angles in a triangle that are close to 90°. Instead, either find small angles first, or use the Cosine rule instead.

Example



$c^2 = 11^2 + 8^2 - 2(11)(8)cos30°$, giving $c = 5.70785$

Now $\frac{sinB}{11} = \frac{sin30°}{5.70785} \Rightarrow sinB = 0.96359 \Rightarrow B = 74.5°$ or $105.5°$

But $\frac{sinA}{8} = \frac{sin30°}{5.70785} \Rightarrow sinA = 0.70079$

$\Rightarrow A = 44.5°$ (not $180 - 44.5$)

$\Rightarrow B = 180 - 30 - 44.5 = 105.5°$

(4) To solve eg $\sin(2x-60°)=0.5$; $0\leq x\leq 360°$:

Let $u=2x-60°$ and note that $-60°\leq u\leq 660°$

Having found the solutions for u (such that $-60°\leq u\leq 660°$), the solutions for x are obtained from $x=\frac{1}{2}(u+60)$.

(5) Starting with $cos^2\theta+sin^2\theta=1$ (A) and

$cos^2\theta-sin^2\theta=cos2\theta$ (B),

$\frac{1}{2}[(A)+(B)]\Rightarrow cos^2\theta=\frac{1}{2}(1+cos2\theta)$

and $\frac{1}{2}[(A)-(B)]\Rightarrow sin^2\theta=\frac{1}{2}(1-cos2\theta)$

(6)(a) In order for $y=arcsinx$ (or $sin^{-1}x$) to be a function, the range of the inverse of $y=sinx$ is restricted to $[-\frac{\pi}{2},\frac{\pi}{2}]$

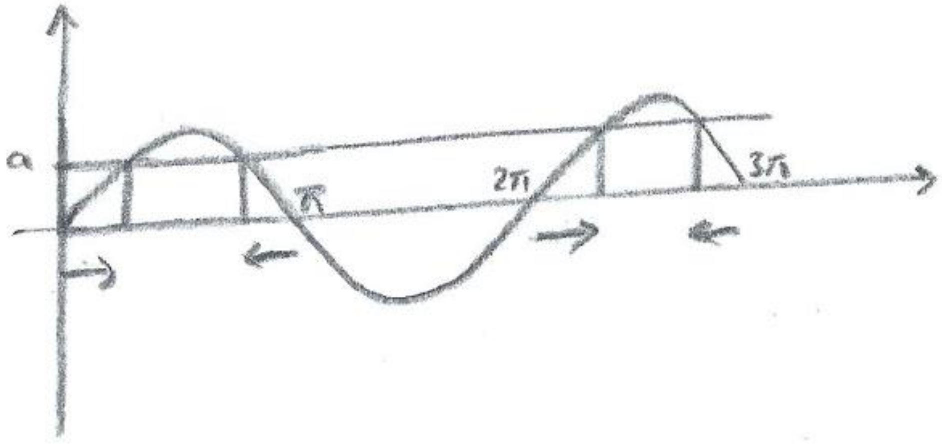
(To avoid vertical duplication for $y=arcsinx$, we ensure that there is no horizontal duplication for $y=sinx$.)

Then $sinx=a\Rightarrow$

$x=arcsin(a)+n(2\pi)$ or $\pi-arcsin(a)+n(2\pi)$ for $n\in\mathbb{Z}$

Alternatively, $x=n\pi+(-1)^n arcsin(a)$

[For even multiples of π, we go forward along the curve, and for odd multiples we go back - see the diagram below.]



(b) In order for $y = arctanx$ (or $tan^{-1}x$) to be a function, the range of the inverse of $y = tanx$ is also restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Then $tanx = a \Rightarrow x = arctan(a) + n\pi$ for $n \in \mathbb{Z}$

(c) In order for $y = arccosx$ (or $cos^{-1}x$) to be a function, the range of the inverse of $y = cosx$ is restricted to $[0, \frac{\pi}{2}]$

(avoiding horizontal duplication for $y = cosx$)

Then $cosx = a \Rightarrow$

$x = arccos(a) + n(2\pi)$ or $2\pi - arccos(a) + n(2\pi)$ for $n \in \mathbb{Z}$

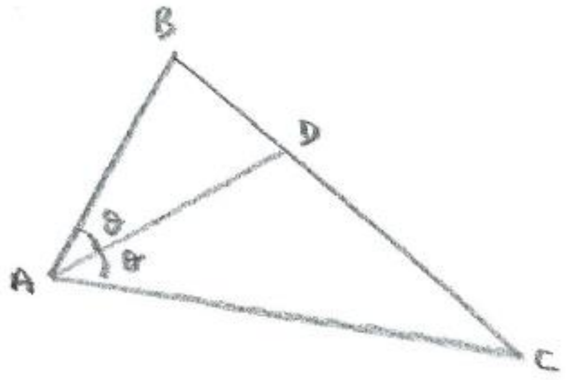
[The 2nd option can also be written as $-arccos(a) + n'(2\pi)$]

Alternatively, $x = 2n\pi \pm arccos(a)$

(7) Angle Bisector Theorem

Referring to the diagram below, the Angle Bisector theorem says that

$\frac{BD}{DC} = \frac{AB}{AC}$



Proof

Method 1

By the Sine rule for triangle ABD, $\frac{BD}{sin\theta} = \frac{AB}{sinADB}$ (1)

and, for triangle ADC, $\frac{DC}{sin\theta} = \frac{AC}{sinADC} = \frac{AC}{sinADB}$ (2)

Then (1) $\Rightarrow \frac{sin\theta}{sinADB} = \frac{BD}{AB}$ and (2) $\Rightarrow \frac{sin\theta}{sinADB} = \frac{DC}{AC}$

so that $\frac{BD}{AB} = \frac{DC}{AC}$

and hence $\frac{BD}{DC} = \frac{AB}{AC}$

Method 2

Area of triangle ABD ÷ Area of triangle ADC = $\frac{\frac{1}{2}AB.ADsin\theta}{\frac{1}{2}AC.ADsin\theta} = \frac{AB}{AC}$

Also,

Area of triangle ABD ÷ Area of triangle ADC = $\frac{\frac{1}{2}BD.ADsinBDA}{\frac{1}{2}AD.DCsinADC} = \frac{BD}{DC}$,

as $\angle BDA = 180 - \angle ADC$, so that $sinBDA = sinADC$

Hence, $\frac{AB}{AC} = \frac{BD}{DC}$