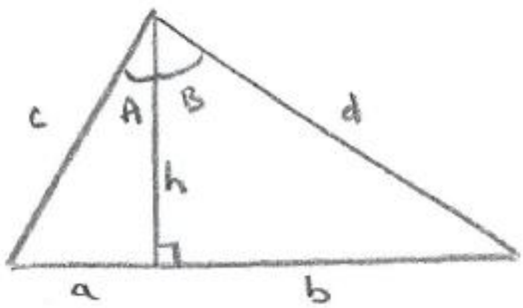
Trigonometry – Compound Angle Formulae

(4 pages; 15/4/21)

(1) Proof of $sin(A+B) = sinAcosB + cosAsinB$



Referring to the diagram,

the area of the large triangle can be formed in two ways:

$\frac{1}{2}cdsin(A+B) = \frac{1}{2}ah + \frac{1}{2}bh$

$\Rightarrow sin(A+B) = \frac{a}{c}.\frac{h}{d} + \frac{b}{d}.\frac{h}{c} = sinAcosB + sinBcosA$

The other compound angle formulae can be derived from the formula for $sin(A+B)$, as follows:

$sin(A-B) = sin(A+[-B]) = sinAcos(-B) + cosAsin(-B)$

$= sinAcosB - cosAsinB$

$cos(A+B) = sin(90-[A+B]) = sin([90-A]-B)$

$= sin(90-A)cosB - cos(90-A)sinB$

$= cosAcosB - sinAsinB$

$cos(A-B) = cos(A+[-B]) = cosAcos(-B) - sinAsin(-B)$

$= cosAcosB + sinAsinB$

$tan(A+B) = \frac{sin(A+B)}{cos(A+B)} = \frac{sinAcosB+cosAsinB}{cosAcosB-sinAsinB}$

$= \frac{tanA+tanB}{1-tanAtanB}$ (after dividing top and bottom by $cosAcosB$)

and similarly for $tan(A-B) = \frac{tanA-tanB}{1+tanAtanB}$

(2) Proof using matrices

Rotation of θ followed by rotation of ϕ

$$\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} cos\phi & -sin\phi \\ sin\phi & cos\phi \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x'' \\ y'' \end{pmatrix}$$

$$\begin{pmatrix} cos\phi & -sin\phi \\ sin\phi & cos\phi \end{pmatrix} \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x'' \\ y'' \end{pmatrix}$$

Also $\begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x'' \\ y'' \end{pmatrix}$

Hence $\cos(\theta+\phi) = cos\theta cos\phi - sin\theta sin\phi$

& $\sin(\theta+\phi) = cos\theta sin\phi + sin\theta cos\phi$ or $sin\theta cos\phi + cos\theta sin\phi$

(3) Expressions for $cos^2\theta$ and $sin^2\theta$

Using the results $cos^2\theta + sin^2\theta = 1$

(by applying Pythagoras to a right-angled triangle with sides $cos\theta, sin\theta$ & 1) and $cos^2\theta - sin^2\theta = \cos(2\theta)$ (from the Compound angle formulae) we can derive:

$cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$ & $sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$

(4) $\boldsymbol{Rsin(\theta + \alpha)}$ and $\boldsymbol{Rcos(\theta - \alpha)}$

The compound angle formulae can be used to write the expression $asin\theta + bcos\theta$ in either of the forms $Rsin(\theta + \alpha)$ or $Rcos(\theta - \alpha)$.

Example

$\sqrt{3}sin\theta + cos\theta = Rsin(\theta + \alpha) = Rsin\theta cos\alpha + Rcos\theta sin\alpha$

and equating coefficients of $sin\theta$ & $cos\theta$ (given that the two expressions are to be equal for all values of θ),

$\sqrt{3} = Rcos\alpha$ & $1 = Rsin\alpha$,

so that $3 + 1 = R^2(cos^2\alpha + sin^2\alpha) = R^2$,

giving $R = 2$ (for example); and $tan\alpha = \frac{1}{\sqrt{3}}$, giving $\alpha = \frac{\pi}{6}$

Thus $\sqrt{3}sin\theta + cos\theta = 2sin\left(\theta + \frac{\pi}{6}\right)$

Alternatively,

$\sqrt{3}sin\theta + cos\theta = Rcos(\theta - \alpha) = Rcos\theta cos\alpha + Rsin\theta sin\alpha$,

so that $\sqrt{3} = Rsin\alpha$ & $1 = Rcos\alpha$, giving $R = 2$ again,

and $tan\alpha = \sqrt{3}$, so that $\alpha = \frac{\pi}{3}$

Thus $\sqrt{3}sin\theta + cos\theta$ can be written as either

$2sin\left(\theta+\frac{\pi}{6}\right)$ or $2\cos\left(\theta-\frac{\pi}{3}\right)$ [see note (b) below]

Had we chosen $Rsin(\theta-\alpha)$ instead, then

$\sqrt{3}sin\theta+cos\theta\ =Rsin\theta cos\alpha-Rcos\theta sin\alpha,$

giving $\sqrt{3}=Rcos\alpha$ & $1=-Rsin\alpha,$

so that $R=2$ and $tan\alpha=-\frac{1}{\sqrt{3}}$, or $\tan(-\alpha)=\frac{1}{\sqrt{3}}$,

with the same result as before.

Similarly,

$\sqrt{3}sin\theta+cos\theta=Rcos(\theta+\alpha)=Rcos\theta cos\alpha-Rsin\theta sin\alpha,$

giving $\sqrt{3}=-Rsin\alpha$ & $1=Rcos\alpha,$

so that $R=2$ and $tan\alpha=-\sqrt{3}$, or $\tan(-\alpha)=\sqrt{3}$, as before

Notes

(a) $Rsin(\theta-\alpha)$ can be chosen, if b is negative; similarly for $Rcos(\theta+\alpha)$ if a is negative.

(b) As $sin(\theta+\alpha)=\cos(90°-[\theta+\alpha])=cos([90°-\alpha]-\theta)$

$=cos(\theta-[90°-\alpha])$, one form can be obtained from the other.

Similarly, $cos(\theta-\alpha)=\cos(\alpha-\theta)=sin(90°-[\alpha-\theta])$

$=sin(\theta+[90°-\alpha])$