Turning Points & Points of Inflexion (7 pages; 5/6/23)

(1) A necessary and sufficient condition for a turning point is that the first non-zero derivative of the function must be of even order (≥ 2).

See the Appendix for a sketch of a proof of this.

The sign of this derivative then determines whether it is a maximum (if negative) or minimum (if positive). Thus, in the case of $y = x^4$ at $x = 0$, $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = 0$, $\frac{d^3y}{dx^3} = 0$ & $\frac{d^4y}{dx^4} = 24$

Thus a necessary (but not sufficient) condition for a turning point is that $\frac{dy}{dx} = 0$.

(2) $\frac{d^2y}{dx^2} \neq 0$ is a sufficient (but not necessary) condition for a turning point (eg for $y = x^2, \frac{d^2y}{dx^2} = 2$)

If $\frac{d^2y}{dx^2} = 0$ (eg for $y = x^4$), then the pattern of $\frac{dy}{dx}$ about the point ($x = 0$ in this case) can be examined, as an alternative to investigating higher derivatives.

(3) Note that the maximum or minimum that occurs at a turning point is only a local maximum or minimum, and the greatest or least value of a function can occur without $\frac{dy}{dx} = 0$ being necessary, if the domain of the function is limited, and the greatest or least value occurs at one end of the domain.

(4) The turning point of a quadratic is midway between its roots.

(5) A polynomial function of the form $(x-a)^{2m}g(x)$, where $m > 0$, has a turning point at $(a, 0)$.

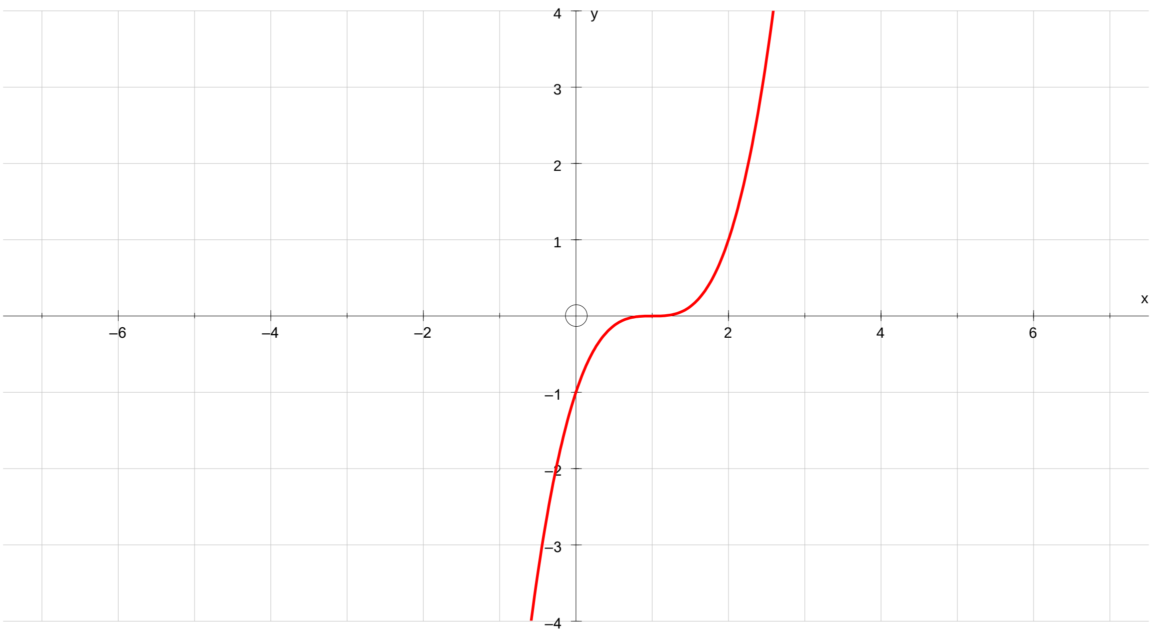
(6) To find the turning points of $y = \frac{x^2-2x+2}{x^2-3x-4}$, consider the quadratic $\frac{x^2-2x+2}{x^2-3x-4} = k$, with $b^2 - 4ac = 0$ (to give a quadratic in k).

Points of Inflexion

(1) A point of inflexion occurs at a turning point of the gradient.

A turning point of a function occurs when the gradient of the function changes sign (either from positive to negative, in the case of a maximum, or from negative to positive, in the case of a minimum). So a point of inflexion occurs when the gradient of the gradient changes sign; ie when $\frac{d^2y}{dx^2}$ changes sign. This is when a function changes from being convex to concave (or vice-versa). (See separate note "Convex & concave functions".)

(2) Example 1: $y = (x-1)^3$



From left to right: $\frac{dy}{dx}$ is positive, falls to zero; then increases again; ie the gradient reaches a minimum (of zero).

Example 2: $y = (1-x)^3$

From left to right: $\frac{dy}{dx}$ is negative, rises to zero; then decreases again; ie the gradient reaches a maximum (of zero).

Example 3: $y = \frac{1}{3}x^3 - x^2 + 2x$

From left to right: $\frac{dy}{dx}$ is positive, falls to 1 (at $x = 1$); then increases again; ie the gradient reaches a (non-zero) minimum.

Thus there is a point of inflexion at $x = 1$, which isn't a stationary point.

[This function was created as follows:

If $\frac{dy}{dx} = (x-1)^2 + 1$, then $\frac{dy}{dx}$ will have a minimum of 1 at $x = 1$;

y is then obtained by expanding and integrating $\frac{dy}{dx}$]

(3) Because a point of inflexion is a turning point of the gradient:

(i) A necessary (but not sufficient) condition for a point of inflexion (turning point of the gradient) is that $\frac{d^2y}{dx^2} = 0$

(e.g. $\frac{d^2y}{dx^2} = 0$ at $x = 0$ for $y = x^4$, but there is no point of inflexion)

(ii) Sufficient (but not necessary) conditions are $\frac{d^2y}{dx^2} = 0$ & $\frac{d^3y}{dx^3} \neq 0$ (eg $y = x^5$, which has a point of inflexion at at $x = 0$, but $\frac{d^3y}{dx^3} = 0$)

Note that a point of inflexion need not be a stationary point (ie where $\frac{dy}{dx} = 0$); eg $y = sinx$ at $x = 0$

(4) Because a point of inflexion is a turning point of the gradient:

A necessary and sufficient condition for a point of inflexion is that the first non-zero derivative of the function is of odd order (≥ 3).

Thus, in the case of $y = x^5 + x$ at $x = 0$,

$\frac{dy}{dx} = 1$, $\frac{d^2y}{dx^2} = 0$, $\frac{d^3y}{dx^3} = 0$, $\frac{d^4y}{dx^4} = 0$, $\frac{d^5y}{dx^5} = 120$

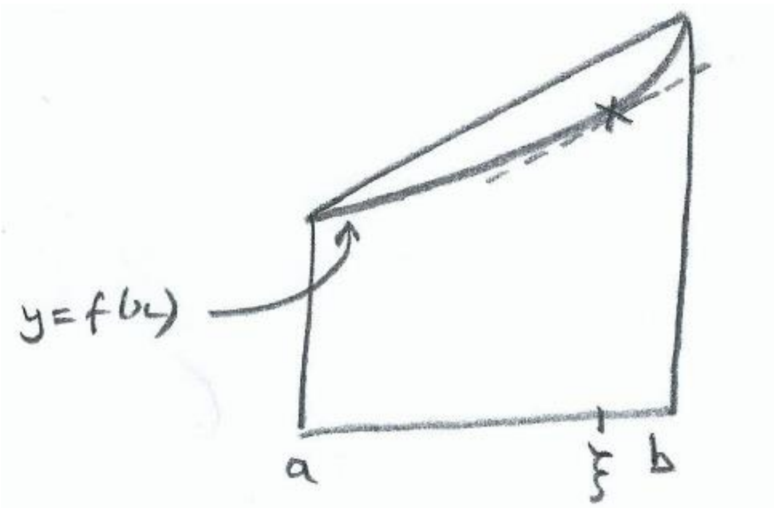
(5) A polynomial function of the form $(x - b)^{2n+1}h(x)$, where $n > 0$, has a point of inflexion at $(b, 0)$.

Appendix: A necessary and sufficient condition for a turning point is that the first non-zero derivative of the function must be of even order (≥ 2).

Sketch of proof

(A) The Mean Value theorem

This states that "If $f(x)$ has a derivative for all values of x in the interval (a, b), then there is a value ξ of x between a and b, such that $f'(\xi) = \frac{f(b)-f(a)}{b-a}$ "



The diagram demonstrates this: the gradient of the curve at

$x = \xi$, ie $f'(\xi)$ equals the gradient of the line (ie $\frac{f(b)-f(a)}{b-a}$).

The theorem can be written in the form

$f(x + h) = f(x) + hf'(x + \theta h)$, where $0 < \theta < 1$ (1)

(with x replacing a and $h = b - a$)

(B) The General Mean Value theorem

It can be shown that (1) extends to

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots \frac{h^n}{n!}f^{(n)}(x+\theta h), \quad (2)$$

where $0 < \theta < 1$

[See, for example, "A course of Pure Mathematics" by G H Hardy (CUP 1933)]

(C) If the first $n-1$ derivatives of $f(x)$ are zero, then

$$(2) \Rightarrow f(x+h) - f(x) = \frac{h^n}{n!}f^{(n)}(x+\theta h)$$

If n is even, then $f(x+h) - f(x) > 0$ if $f^{(n)}(x+\theta h) > 0$

This is true for positive and negative h, so that there is a local minimum at x.

Also (again with even n), $f(x+h) - f(x) < 0$ if $f^{(n)}(x+\theta h) < 0$, and then there is a local maximum at x.

If instead n is odd, then the sign of $\frac{h^n}{n!}f^{(n)}(x+\theta h)$ will change as h changes from negative to positive, so that there will not be a turning point.

Thus a necessary and sufficient condition for a turning point is that the first non-zero derivative of the function must be of even order (≥ 2).