TMUA Exercises – Series - Sol'ns (3 pages; 2/11/22)

(1) Show that $\sum_{r=0}^{n} \binom{n}{r} = 2^n$

Solution

Method 1: Consider $(1+1)^n$

Method 2: Pascal's triangle

The sum of each row is twice the sum of the previous one.

eg $1+5+10+10+5+1$

$=(1+10+5)[alternate\ terms]+(5+10+1)$

$=2(1+10+5)=2(1+[4+6]+[4+1])$

& $1+6+15+20+15+6+1$

$=(1+15+15+1)+(6+20+6)$

$=(1+[5+10]+[10+5]+1)$

$+([1+5]+[10+10]+[5+1])$

Method 3: Counting ways of selecting any number of items

1st counting method: $\sum_{r=0}^{n}\binom{n}{r}$

2nd counting method: For each object, there are 2 choices: include or exclude; giving 2^n

[Note: 1 way of choosing no objects is included in the total.]

Method 4: Induction

If true for $n=k$, so that $\sum_{r=0}^{k}\binom{k}{r}=2^k$,

then $\sum_{r=0}^{k+1}\binom{k+1}{r}=\binom{k+1}{0}+\{\sum_{r=1}^{k}\binom{k+1}{r}\}+\binom{k+1}{k+1}$

$=1+\sum_{r=1}^{k}\left\{\binom{k}{r-1}+\binom{k}{r}\right\}+1$

$=1+\{\sum_{r-1=0}^{k-1}\binom{k}{r-1}\}+[\{\sum_{r=0}^{k}\binom{k}{r}\}-\binom{k}{0}]+1$

$= 1 + \{\sum_{R=0}^{k-1} \binom{k}{R}\} + [2^k - 1] + 1$

$= 1 + \{\sum_{R=0}^{k} \binom{k}{R}\} - \binom{k}{k} + 2^k$

$= 1 + 2^k - 1 + 2^k$

$= 2^{k+1}$