TMUA Specimen Paper 1 Solutions (8 pages; 30/12/21)

Q1

$x-3y+1=0$ & $3x^2-7xy=5 \Rightarrow 3x^2-7x\left(\frac{x+1}{3}\right)=5$

$\Rightarrow 9x^2-7x(x+1)=15$

$\Rightarrow 2x^2-7x-15=0$

The sum of the two roots is $\frac{-(-7)}{2}=3.5$

Answer: D

Q2

$sin^2\theta+3cos\theta=3 \Rightarrow (1-cos^2\theta)+3cos\theta=3$

$\Rightarrow cos^2\theta-3cos\theta+2=0$

$\Rightarrow (cos\theta-1)(cos\theta-2)=0$

$\Rightarrow cos\theta=1$ (rejecting $cos\theta=2$)

Given that $0\leq\theta\leq 4\pi$, the possible solutions are:

$\theta=0, 2\pi$ & 4π

Answer: D

Q3

The midpoint of the line segment joining $(2,-6)$& $(5,4)$ is

$(\frac{7}{2}, -1)$, and the gradient of this line segment is $\frac{4-(-6)}{5-2}=\frac{10}{3}$

So the equation of the perpendicular bisector is

$\frac{y-(-1)}{x-\frac{7}{2}} = -\frac{3}{10}$

Then when $y = 0, x - \frac{7}{2} = -\frac{10}{3}$, so that $x = \frac{21-20}{6} = \frac{1}{6}$

Answer: B

Q4

$(x^2 - 1)(x - 2) > 0 \Leftrightarrow (x - 1)(x + 1)(x - 2) > 0$

or $(x + 1)(x - 1)(x - 2) > 0$; with critical points at $x = -1, 1 \;\&\; 2$

LHS is cubic, with large negative value for large negative value of x. Then positive for $-1 < x < 1$, negative for $1 < x < 2$, and positive for $2 < x$.

So required set of values is $-1 < x < 1$ and $2 < x$.

Answer: E

Q5

$y = -log_{10}(1 - x) = log_{10}(\frac{1}{1-x})$

$\Rightarrow 10^y = \frac{1}{1-x}$

$\Rightarrow 1 - x = 10^{-y}$

$\Rightarrow x = 1 - 10^{-y}$

Answer: D

Q6

Let $f(x) = x^3 + 4cx^2 + x(c+1)^2 - 6$

Then, as $x + 2$ is a factor, $f(-2) = 0$, so that

$-8 + 16c - 2(c+1)^2 - 6 = 0$

$\Rightarrow -2c^2 + 12c - 16 = 0$ or $c^2 - 6c + 8 = 0$

The sum of the roots is $-\frac{(-6)}{1} = 6$.

Answer: D

Q7

P(Balls are not the same colour)

$= 1 - P$(Balls are the same colour)

$= 1 - 3P$(Both balls are Red)

$= 1 - 3\left(\frac{1}{3}\right)\left(\frac{n-1}{3n-1}\right)$

$= \frac{(3n-1)-(n-1)}{3n-1} = \frac{2n}{3n-1}$

Answer: C

Q8

$a^x b^{2x} c^{3x} = 2 \Rightarrow a^x (b^2)^x (c^3)^x = 2$

$\Rightarrow (ab^2c^3)^x = 2$

$\Rightarrow x log_{10}(ab^2c^3) = log_{10}(2)$

$\Rightarrow x = \frac{log_{10}(2)}{log_{10}(ab^2c^3)}$

Answer: F

Q9

Let the roots be x_1 and x_2 (where $x_1 > x_2$).

Then $x_1 + x_2 = -\frac{(-11)}{2}$ and $x_1 x_2 = \frac{c}{2}$

And $x_1 - x_2 = 2$

Now $(x_1 + x_2)^2 - (x_1 - x_2)^2 = 4x_1 x_2$,

so that $\frac{121}{4} - 4 = 2c$, and so $c = \frac{1}{8}(121 - 16) = \frac{105}{8}$

Answer: A

Q10

Reflecting $y = f(x)$ in the line $y = b$ can be shown to give $2b - y = f(x)$ [reflecting $y = f(x)$ in the line $x = a$ gives

$y = f(2a - x)$]

Proof: The reflection in $y = b$ is equivalent to a translation of $\begin{pmatrix} 0 \\ -b \end{pmatrix}$, followed by a reflection in the x-axis, and then a translation of $\begin{pmatrix} 0 \\ b \end{pmatrix}$: this produces $y = f(x) \to y = f(x) - b \to -(f(x) - b)$

$\to -(f(x) - b) + b = 2b - f(x)$

When $f(x) = cosx$ and $b = 1$ this gives $y = 2 - cosx$

Then a translation of $\begin{pmatrix} \frac{\pi}{4} \\ 0 \end{pmatrix}$ gives $y = 2 - \cos\,(x - \frac{\pi}{4})$,

Answer: D

Q11

Let $y = 2^x$, so that $y^2 - 8y + 15 = 0$

and $(y-3)(y-5) = 0$

$\Leftrightarrow 2^x = 3$ or 5

Sum of roots is $log_2 3 + log_2 5$

$= \frac{log_{10}3}{log_{10}2} + \frac{log_{10}5}{log_{10}2} = \frac{log_{10}15}{log_{10}2}$

Answer: E

Q12

Volume $= \frac{1}{2}(2x)\left(x\sqrt{3}\right)d = x^2 d\sqrt{3}$

Surface Area = surface area of prism excluding ends + area of 2 ends

$= 3(2x)d + 2 \times \frac{1}{2}(2x)\left(x\sqrt{3}\right) = 6xd + 2x^2\sqrt{3}$

Volume = Surface Area $\Rightarrow x^2 d\sqrt{3} = 6xd + 2x^2\sqrt{3}$

$\Rightarrow d\left(x^2\sqrt{3} - 6x\right) = 2x^2\sqrt{3}$

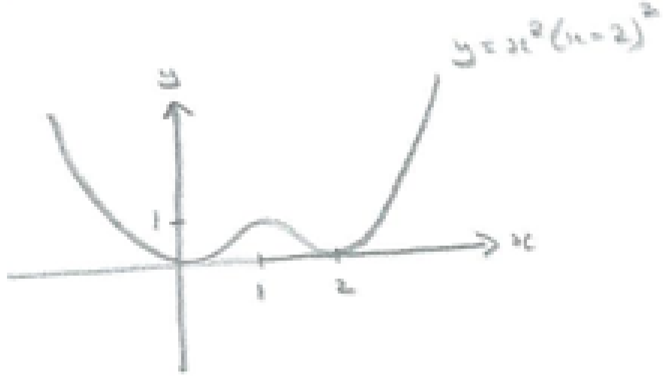
$\Rightarrow d = \frac{2x\sqrt{3}}{x\sqrt{3}-6} = \frac{2x}{x-2\sqrt{3}}$

Answer: D

Q13

Equivalently, consider the roots of $x^2(x^2 - 4x + 4) = 10$

ie $x^2(x-2)^2 = 10$



Referring to the graph, there are 2 roots.

Answer: C

Q14

A: $blogy = xloga$ - not the required straight line

B: $logy = loga + xlogb$ - not the required straight line

C: $2logy = log(a + x^b)$ - not the required straight line

D: $logy = loga + blogx$, which is the required straight line

Answer: D

Q15

$\int_0^1 (x-a)^2 dx = \int_0^1 x^2 - 2ax + a^2 dx$

$= \left[\frac{1}{3}x^3 - ax^2 + a^2x\right]_0^1 = \frac{1}{3} - a + a^2$

$= (a - \frac{1}{2})^2 - \frac{1}{4} + \frac{1}{3}$

The smallest value is therefore $-\frac{1}{4} + \frac{1}{3} = \frac{1}{12}$

Answer: A

Q16

$\frac{10^{c-2d}\times 20^{2c+d}}{8^c\times 125^{c+d}} = \frac{2^{c-2d}5^{c-2d}2^{2(2c+d)}5^{2c+d}}{2^{3c}5^{3(c+d)}}$

$= 2^{c-2d+4c+2d-3c}5^{c-2d+2c+d-3c-3d}$

$= 2^{2c}5^{-4d}$

This will be an integer when $c \geq 0$ and $d \leq 0$.

As c and d are non-zero integers, this condition becomes

$c > 0$ and $d < 0$.

Answer: E

Q17

There will be real distinct roots when the discriminant of

$ax^2 + (a-2)x - 2 = 0$ is positive;

ie when $(a-2)^2 - 4a(-2) > 0$

$\Leftrightarrow a^2 - 4a + 4 + 8a > 0$

ie $a^2 + 4a + 4 > 0$

or $(a+2)^2 > 0$

Hence required condition is $a \neq -2$.

Answer: D

Q18

$\sin(2x) = 0.5$ when $2x = \frac{\pi}{6}$ & $\pi - \frac{\pi}{6}$; ie when $x = \frac{\pi}{12}$ & $\frac{5\pi}{12}$

So, considering the graph of $y = \sin(2x)$,

$\sin(2x) \geq 0.5$ for $\frac{\pi}{12} \leq x \leq \frac{5\pi}{12}$

In this interval, $tanx > 0$, and $tanx \leq 1$ for $x \leq \frac{\pi}{4}$, so that the interval for which $-1 \leq tanx \leq 1$ and $\sin(2x) \geq 0.5$ is

$\frac{\pi}{12} \leq x \leq \frac{\pi}{4}$, which has length $\frac{3\pi}{12} - \frac{\pi}{12} = \frac{\pi}{6}$

Answer: B

Q19

$4r - 4 = 4r^3 - 4r$

$\Rightarrow r^3 - 2r + 1 = 0$

$\Rightarrow (r - 1)(r^2 + r - 1) = 0$

$r \neq 1$, and $r^2 + r - 1 = 0 \Rightarrow r = \frac{-1\pm\sqrt{5}}{2}$

So, as $r > 0, r = \frac{-1+\sqrt{5}}{2}$

and $S_\infty = \frac{4}{1-(\frac{-1+\sqrt{5}}{2})} = \frac{8}{2+1-\sqrt{5}} = \frac{8(3+\sqrt{5})}{9-5} = 2(3 + \sqrt{5})$

Answer: D

Q20

Required coefficient is

4(coefficient of x^2 in $6(2x + 3x^2) + 15(2x + 3x^2)^2$)

$-$ constant term in $(1 - 1)$

$= 4\big(18 + 15(4)\big) = 312$

Answer: G