TMUA Practice Paper 1 Solutions (13 pages; 1/2/22)

[aka 2016 paper]

Q1

Equating constant terms : $b^3 = -3\sqrt{3} \Rightarrow b = -\sqrt{3}$

Equating coefficients of x^3: $a^3 = 8 \Rightarrow a = 2$

Equating coefficients of x^2 : $3a^2b = -p$

$\Rightarrow p = -3(4)\left(-\sqrt{3}\right) = 12\sqrt{3}$

Answer : H

Q2

Writing $f(x) = 3x^3 + 13x^2 + 8x + a$,

$(x+2)$ is a factor $\Rightarrow f(-2) = 0$,

so that $-24 + 52 - 16 + a = 0 \Rightarrow a = -12$

So answer must be D or E.

If answer is D, then $(x-3)$ is a factor $\Rightarrow f(3) = 0$,

$f(3) = 3(27) + 13(9) + 8(3) - 12 \neq 0$, so answer must be E.

Answer : E

Q3

$y = 2x^{-2} \Rightarrow \frac{dy}{dx} = -2(2)x^{-3}$

$\frac{dy}{dx}\ |x = 1\ = -4$

Also, when $x = 1, y = 2$

So eq'n of normal is $\frac{y-2}{x-1} = -\frac{1}{(-4)}$

$\Rightarrow 4(y-2) = x - 1$

At P, $y = 0 \Rightarrow -8 = x - 1 \Rightarrow x = -7$, so that P is $(-7,0)$

At Q, $x = 0 \Rightarrow 4(y-2) = -1 \Rightarrow y = \frac{7}{4}$, so that Q is $(0, \frac{7}{4})$

Then $|PQ|^2 = (-7)^2 + (\frac{7}{4})^2 = \frac{49(16+1)}{16}$, and $|PQ| = \frac{7\sqrt{17}}{4}$

Answer : C

Q4

n odd $\Rightarrow a_n = -1 - 1 - 1 = -3$

n even $\Rightarrow a_n = 1 + 1 + 1 = 3$

$\Rightarrow \sum_{n=1}^{39} a_n = 20(-3) + 19(3) = -3$

Answer : B

Q5

The curve $y = x^2 - 1$ crosses the x-axis at $x = 1$.

By symmetry, required area is

$2(-\int_0^1 x^2 - 1\, dx + \int_1^2 x^2 - 1\, dx)$

$= -2\left[\frac{1}{3}x^3 - x\right]_0^1 + 2\left[\frac{1}{3}x^3 - x\right]_1^2$

$= -2\left(\frac{1}{3} - 1\right) + 2\left(\frac{8}{3} - 2\right) - 2(\frac{1}{3} - 1)$

$= 2 - 4 + 2 + \frac{2}{3}(-1 + 8 - 1) = 4$

Answer : C

Q6

25% is a weighted average of 30% (with weight $\frac{12}{20}$), 20% (with weight $\frac{5}{20}$) and R% (say) (with weight $\frac{3}{20}$);

ie $\frac{12}{20} \times 30 + \frac{5}{20} \times 20 + \frac{3}{20} \times R = 25$

$\Rightarrow 12(30) + 5(20) + 3R = 20(25)$

$\Rightarrow R = \frac{1}{3}(500 - 360 - 100) = \frac{40}{3}$ or $13\frac{1}{3}$

Answer : C

Q7

Without loss of generality, let there be 60 women and 40 men.

Let $C = C_M + C_W$ play cricket, and $T = T_M + T_W$ play tennis.

$P(W\&T) = P(W)P(T\,|W) = 0.6\,P(T\,|W) = 0.6\,\frac{T_W}{60}$

Then $C_M = \frac{2}{5}(40) = 16$ and $C_W = \frac{2}{3}\,C$

Hence $C = 16 + \frac{2}{3}C$, so that $16 = \frac{C}{3}$ and $C = 48$

So $C_M = 16$ and $C_W = \frac{2}{3}(48) = 32$

Then $T_W + C_W = 60$, so that $T_W = 28$

Then $P(W\&T) = 0.6\,\frac{T_W}{60} = 0.6\left(\frac{28}{60}\right) = \frac{3}{5}\left(\frac{7}{15}\right) = \frac{7}{25}$

Answer : B

Q8

Let $y = sin(2x)$, so that

$1 - y^2 + \sqrt{3}y - \frac{7}{4} = 0$

and $4y^2 - 4\sqrt{3}y + 3 = 0$

$\Rightarrow y = \frac{4\sqrt{3} \pm \sqrt{48-48}}{8} = \frac{\sqrt{3}}{2}$

So $sin(2x) = \frac{\sqrt{3}}{2}$, with $0 \leq x \leq 360$

Let $u = 2x$, so that $sinu = \frac{\sqrt{3}}{2}$, with $0 \leq u \leq 720$

$\Rightarrow u = 60, 120, 60 + 360, 120 + 360$

So maximum x is $\frac{1}{2}(120 + 360) = 240$

Answer : F

Q9

Centre of original circle is (5,4)

and $r^2 = (5-3)^2 + (4-3)^2 = 5$

So eq'n is $(x-5)^2 + (y-4)^2 = 5$

Translation of $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ produces $([x+3]-5)^2 + (y-4)^2 = 5$

Reflection in x-axis produces $(x-2)^2 + (-y-4)^2 = 5$

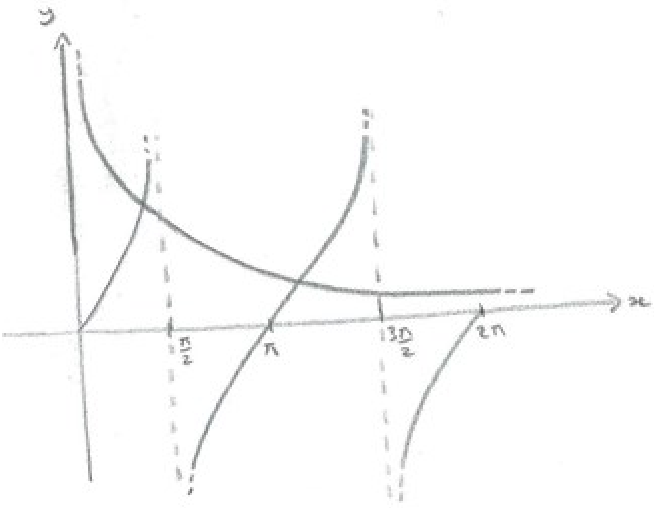
The enlargement produces $(x-2)^2 + (-y-4)^2 = 5(4^2)$

ie $(x-2)^2 + (y+4)^2 = 80$

Answer : D

Q10

Rewrite the eq'n as $tanx = \frac{1}{x}$



Referring to the diagram, there are 2 sol'ns to this eq'n in the interval $0 < x \leq 2\pi$.

So, by symmetry, and as $x = 0$ is not a sol'n, there are 4 sol'ns in the interval $-2\pi \leq x \leq 2\pi$

Answer : E

Q11

If $y = 2^{2x}$, then $4^{2x} = (2^2)^{2x} = 2^{2(2x)} = y^2$,

and $4^{2x} + 12 = 2^{2x+3}$ becomes $y^2 - 8y + 12 = 0$

$\Rightarrow (y - 2)(y - 6) = 0$

so that $2^{2q} = 2$ and $2^{2p} = 6$

Hence $q = \frac{1}{2}$ and $p = \frac{1}{2}log_2 6 = \frac{1}{2}\frac{log_{10}6}{log_{10}2}$,

and $p - q = \frac{1}{2}(\frac{log_{10}6}{log_{10}2} - 1)$

$= \frac{log_{10}6 - log_{10}2}{2log_{10}2} = \frac{log_{10}3}{log_{10}4}$

Answer : E

Q12

Let $2d\ and\ 2h$ be the width and height of the cylinder (as it appears in the diagram in the question)

Volume of cylinder, $V = 2h(\pi d^2)$

and (from the diagram in the question) $d^2 + h^2 = 5^2$

Thus $V = 2h\pi(25 - h^2) = 50\pi h - 2\pi h^3$

$\frac{dV}{dh} = 50\pi - 6\pi h^2$

$\frac{dV}{dh} = 0 \Rightarrow 50 = 6h^2 \Rightarrow h = \sqrt{\frac{50}{6}}$ and $d^2 = 25 - \frac{50}{6} = \frac{100}{6} = \frac{50}{3}$

Then $V = 2\sqrt{\frac{50}{6}}\,\pi\left(\frac{50}{3}\right) = 2(5)\sqrt{\frac{2}{6}}\,\pi\left(\frac{50}{3}\right) = \frac{500\pi}{3\sqrt{3}} = \frac{500\sqrt{3}}{9}\pi$

Answer : E

Q13

Writing $f(x) = 3x^5 - 10x^3 - 120x + 30$,

$f'(x) = 15x^4 - 30x^2 - 120$

Then $f'(x) = 0 \Rightarrow (x^2 - 4)(x^2 + 2) = 0 \Rightarrow x = \pm 2$

$f''(x) = 60x^3 - 60x$

$f''(-2) < 0 \Rightarrow$ maximum

$f''(2) > 0 \Rightarrow$ minimum

(and these are the only two turning points).

[The shape of a quintic means that, if $f'(x) = 0$ and $f''(x) \neq 0$ for $x = -2$ and $x = 2$, then there would have to be a maximum at $x = -2$ and a minimum at $x = 2$.]

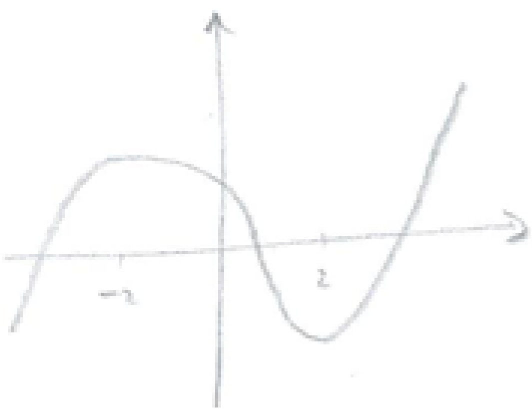
$f(-2) = 3(-32) - 10(-8) - 120(-2) + 30$

$= -96 + 80 + 240 + 30 > 0$

and $f(2) = 3(32) - 10(8) - 120(2) + 30$

$= 96 - 80 - 240 + 30 < 0$

so that the graph of $f(x)$ has the shape shown in the diagram below, and therefore there are 3 real roots of $f(x) = 0$



Answer : C

Q14

Write $T: 4, 4r, 4r^2$ ... and $U: 4, 4R, 4R^2$ ...

Then $4r + 4R = 3$ (1) and $4r^2 + 4R^2 = \frac{5}{4}$ (2)

And $S_\infty = T_\infty + U_\infty = \frac{4}{1-r} + \frac{4}{1-R}$

From (1), $R = \frac{3-4r}{4}$

Substituting into (2):

$16r^2 + (3-4r)^2 = 5$

$\Rightarrow 32r^2 - 24r + 4 = 0$

$\Rightarrow 8r^2 - 6r + 1 = 0$

$\Rightarrow (4r-1)(2r-1) = 0$

$\Rightarrow r = \frac{1}{4}\ or\ \frac{1}{2}$

$\Rightarrow R = \frac{1}{2}\ or\ \frac{1}{4}$ (or by symmetry)

So $S_\infty = \frac{4}{(\frac{1}{2})} + \frac{4}{(\frac{3}{4})} = 8 + \frac{16}{3} = \frac{40}{3}$

Answer : D

Q15

$y = (2x+a)(x-2a)^2$

$\frac{dy}{dx} = 2(x-2a)^2 + 2(2x+a)(x-2a)$

When $x = 1, \frac{dy}{dx} = 2(1-2a)^2 + 2(2+a)(1-2a)$

$= 2(1-2a)(1-2a+2+a)$

$= 2(1-2a)(3-a) = A$, say

Then $\frac{dA}{da} = 0 \Rightarrow -4(3-a) + 2(1-2a)(-1) = 0$

$\Rightarrow -14 + 8a = 0 \Rightarrow a = \frac{7}{4}$

When $a = \frac{7}{4}$, $A = 2\left(1 - \frac{7}{2}\right)\left(3 - \frac{7}{4}\right) = \frac{1}{4}(-5)(5) = -\frac{25}{4}$

Answer: C

Q16

$log_{10}2 + log_{10}(y-1) = 2log_{10}x$

$\Rightarrow log_{10}2(y-1) = log_{10}(x^2)$

$\Rightarrow 2(y-1) = x^2$

And $log_{10}(y+3-3x) = 0 \Rightarrow y+3-3x = 1$

$\Rightarrow 3x = y+2$

$\Rightarrow 9x^2 = (y+2)^2$

So $18(y-1) = (y+2)^2$

$\Rightarrow y^2 - 14y + 22 = 0$

$\Rightarrow y = \frac{14 \pm \sqrt{196-88}}{2} = \frac{14 \pm \sqrt{108}}{2} = 7 \pm \sqrt{27} = 7 \pm 3\sqrt{3}$

Answer: C

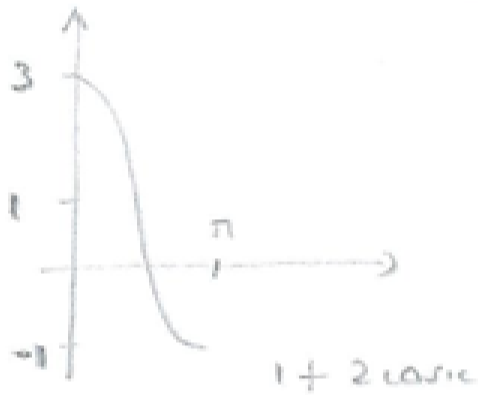
Q17

$y < 0 \Rightarrow$ either $1 + 2cosx > 0$ & $\cos(2x) < 0$ (A)

or $1 + 2cosx < 0$ & $\cos(2x) > 0$ (B)

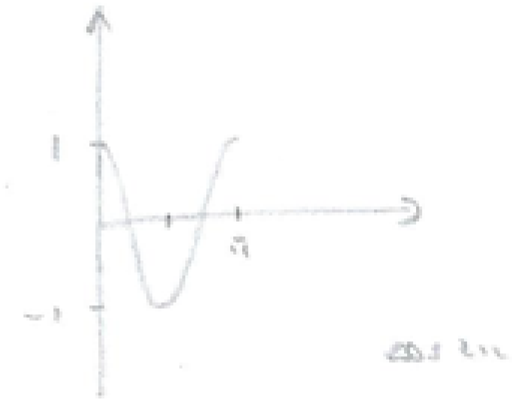
For (A):

$1 + 2cosx = 0 \Rightarrow cosx = -\frac{1}{2} \Rightarrow x = \frac{2\pi}{3}$



Referring to the graph of $1 + 2cosx$,

$1 + 2cosx > 0$ when $0 < x < \frac{2\pi}{3}$ (given that $0 < x < \pi$)



and, referring to the graph of $\cos(2x)$,

$\cos(2x) < 0$ when $\frac{\pi}{4} < x < \frac{3\pi}{4}$

So (A) $\Rightarrow 0 < x < \frac{2\pi}{3}\ and\ \frac{\pi}{4} < x < \frac{3\pi}{4}$; ie $\frac{\pi}{4} < x < \frac{2\pi}{3}$

And (B) $\Rightarrow \frac{2\pi}{3} < x < \pi\ and\ (0 < x < \frac{\pi}{4}\ or\ \frac{3\pi}{4} < x < \pi)$;

ie $\frac{3\pi}{4} < x < \pi$

Answer: D

Q18

$f(x) = (1-x)x^{-\frac{2}{3}} = x^{-\frac{2}{3}} - x^{\frac{1}{3}}$

[Note : $x^{\frac{1}{3}}$ is defined for negative x, but $x^{\frac{1}{2}}$ isn't (except as an imaginary number).]

$f'(x) = -\frac{2}{3}x^{-\frac{5}{3}} - \frac{1}{3}x^{-\frac{2}{3}}$

Then $f'(x) < 0 \Rightarrow 2x^{-\frac{5}{3}} + x^{-\frac{2}{3}} > 0$

$\Rightarrow x^{-\frac{2}{3}}(2x^{-1} + 1) > 0$

Now $x^{-\frac{2}{3}} > 0$ for all $x \neq 0$ [If $x < 0, x^{\frac{1}{3}} < 0$ & $x^{\frac{2}{3}} > 0$, so $x^{-\frac{2}{3}} > 0$]

So when $f(x)$ is decreasing, $2x^{-1} + 1 > 0 \Rightarrow x^{-1} > -\frac{1}{2}$

If $x > 0$, then $1 > -\frac{x}{2} \Rightarrow x > -2$; ie $x > 0$

If $x < 0$, then $1 < -\frac{x}{2} \Rightarrow x < -2$; ie $x < -2$

The Answer is therefore $x < -2, x > 0$, but this isn't one of the options! The official solutions say:

> It is therefore increasing in the region $x < -2$ and $x > 0$. (It it not increasing at $x = -2$, but rather it is stationary at that point.) The correct answer is therefore A (though the question mistakenly says $x \leqslant -2$).

Q19

$(1 + 2x + 3x^2)^6 = 1 + 6(2x + 3x^2) + \binom{6}{2}(2x + 3x^2)^2$

$+ \binom{6}{3}(2x + 3x^2)^3 + \cdots$

(subsequent terms involve powers of x greater than 3)

The coefficient of x^3 in $(1 + 2x + 3x^2)^6$ is therefore:

coefficient of x in $15(2+3x)^2$

$+$ constant term in $20(2+3x)^3$

$= 15(12) + 20(8) = 180 + 160 = 340$

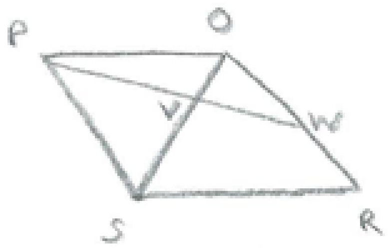
The coefficient of x^4 in $(1-ax^2)^5$ is $\binom{5}{2}(-a)^2 = 10a^2$

So $340 = 2(10a^2) \Rightarrow a = \pm\sqrt{17}$

Answer: B

Q20

The shortest distance can be established by drawing a net of the pyramid, but there are two possible routes. The first one, PW, crossing OS (or OQ), is shown below (note that the angles in the triangles OPS and ORS are maintained).



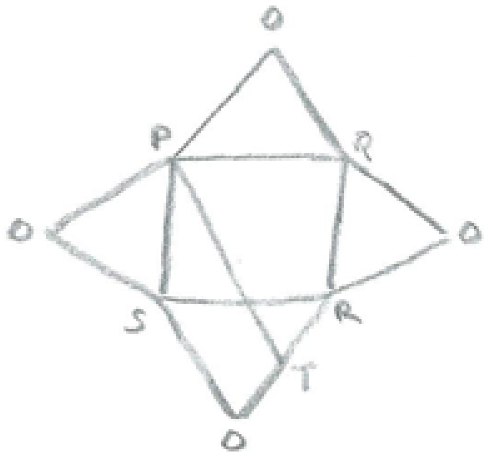
Applying the Cosine rule to OPW,

$PW^2 = PO^2 + OW^2 - 2PO.OWcosPOR$

$= 20^2 + 10^2 - 2(20)(10)cos120°$

$= 500 - 400\left(-\frac{1}{2}\right) = 700$

The second possible route, PT, crossing SR (or RQ), is shown below.



[In the diagram, it looks as though PT bisects SR, but this is not in fact the case.]

Applying the Cosine rule to PTS,

$PT^2 = PS^2 + ST^2 - 2PS.STcosPST$

$= 20^2 + (10\sqrt{3})^2 - 2(20)(10\sqrt{3})cos120°$

$= 700 - 400\sqrt{3}(-\frac{1}{2})$

$= 700 + 200\sqrt{3}$

So, as $PT^2 > PW^2$, the shortest distance is $PW = 10\sqrt{7}$.

Answer: D