TMUA 2021 Paper 2 Solutions (13 pages; 13/11/22)

Q1

$\int_1^4 3\sqrt{x} + \frac{4}{x^2}\ dx = [\frac{3}{\left(\frac{3}{2}\right)}x^{\frac{3}{2}} + \frac{4}{(-1)}x^{-1}]_1^4$

$= \left(2(8) - 4\left(\frac{1}{4}\right)\right) - (2-4) = 17$

Answer: D

Q2

The straight line through B and D passes through the midpoint of AC : (2,1)

and is perpendicular to AC, which has a gradient of $\frac{0-2}{4-0} = -\frac{1}{2}$

So BD has gradient 2, and equation $\frac{y-1}{x-2} = 2$, or $y = 2x - 3$

Answer: E

Q3

I: Not necessary, as eg there could be 8 students wearing glasses in a class of 30

II: Let the number of students in the class be n, and the number who wear glasses be m. Then $\frac{m}{n} = \frac{4}{15}$, so that $n = \frac{15m}{4}$. As n is an integer, and 4 is not a divisor of 15, 4 must be a divisor of m, so that $n = 3(5)\left(\frac{m}{4}\right)$, where $\frac{m}{4}$ is an integer. Thus n is divisible by 3.

III: Not necessary, as eg there could be 4 students wearing glasses in a class of 15

Answer: C

Q4

I: Doesn't provide a counterexample, as a is a factor of bc, and is a factor of b (for example).

II: Does provide a counterexample, as a is a factor of bc, but is not a factor of either $b\ or\ c$.

III: Doesn't provide a counterexample, as a is a factor of bc, and is a factor of c.

Answer: C

Q5

A: Correct

B: Incorrect, as $cosx = -\sqrt{1 - sin^2x}$ is also a solution (noting that $\sqrt{a}$ is defined to be a positive quantity).

Answer: B

Q6

[It is assumed that "3 real roots" means "3 distinct real roots".]

Consider the following 4 examples:

(i) Cubic with 3 real roots, and therefore 2 stationary points (both P & Q true).

(ii) Polynomial with 3 real roots, but more than 2 stationary points (P true, but Q not true).

(iii) Cubic with 1 real root, and 2 stationary points (P not true, but Q true).

(iv) Cubic with 1 real root, and no stationary points (P not true, and Q not true).

So $P \nRightarrow Q$ & $Q \nRightarrow P$.

A can be written as: $Q \Rightarrow P$ & $P \nRightarrow Q$. So, as $Q \Rightarrow P$ is not true, A is not correct.

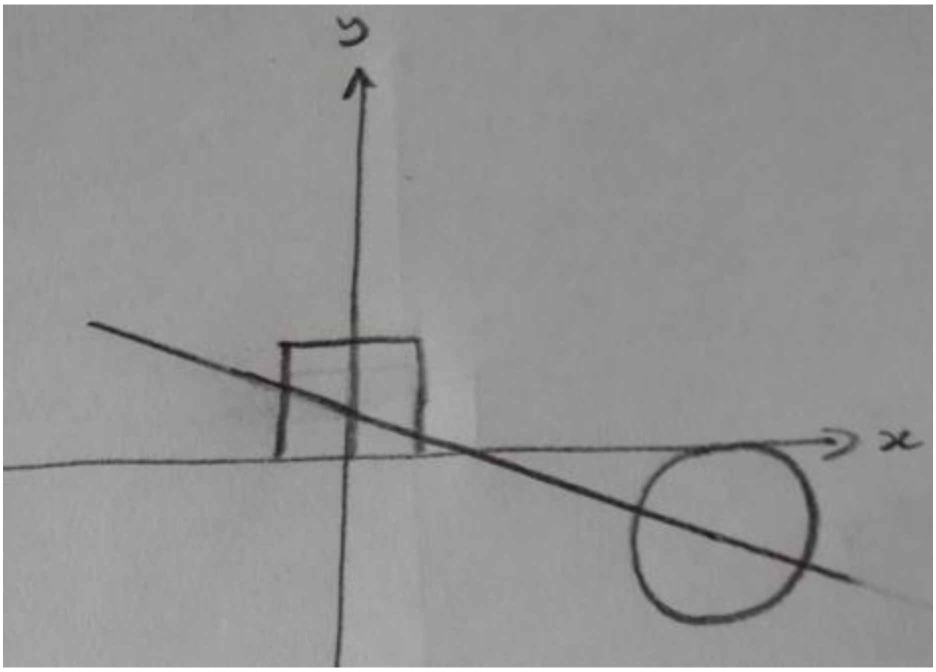
B can be written as: $P \Rightarrow Q$ & $Q \nRightarrow P$. So, as $P \Rightarrow Q$ is not true, B is not correct.

C can be written as: $Q \Rightarrow P$ & $P \Rightarrow Q$. So, as $Q \Rightarrow P$ is not true (for example), C is not correct.

D can be written as: $Q \nRightarrow P$ & $P \nRightarrow Q$. So, as both statements are true, D is correct.

Answer: D

Q7



The straight line must divide the circle into two semicircles, and hence must pass through the centre $(9,-2)$.

[Not so obviously:] The straight line must pass through the centre of the square.

Proof: The line must cross either the two vertical sides or the two horizontal sides, otherwise it will lie to one side of the diagonal, and will not bisect the area of the square.

Without loss of generality, suppose that the line crosses the two vertical sides, and that it does this at $(-1,a)$ and $(1,b)$.

The area below the line is then $\frac{1}{2}(a+b)(2)=a+b$, and as this must equal half of the area of square, $a+b=\frac{1}{2}(2^2)=2$, so that $b=2-a$, and the eq'n of the line is $\frac{y-a}{x-(-1)}=\frac{a-(2-a)}{(-1)-1}$

or $y=a-\frac{(2a-2)(x+1)}{2}=a-(a-1)(x+1)=(1-a)x+1$

Then, when $x=0, y=1$, so that the line passes through the centre of the square, as required.

Thus the line passes through $(9,-2)$ and $(0,1)$, and so its eq'n is:

$\frac{y-1}{x-0}=\frac{-2-1}{9-0}$

and when the line crosses the x-axis, $\frac{-1}{x}=-\frac{1}{3}$, so that $x=3$

Answer: B

Q8

$p(a)=p(b)$ is a sufficient condition: the curve $y=p(x)$ is either a straight line, in which case $p'(x)=0$ for all x between a & b; or

$y = p(x)$ rises above $p(a)$ to achieve a maximum, before falling to $p(b) = p(a)$ (possibly after one or more minima or maxima); or $y = p(x)$ falls below $p(a)$ to achieve a minimum. In each of these cases (*) is satisfied.

$p(a) = p(b)$ isn't a necessary condition: a maximum or minimum could exist between a & b when $p(a) \neq p(b)$

Answer: C

Q9

All cubic graphs cross the x-axis, so I is sufficient.

If $f(x) = x^2 + 1$, then $f'(0) = 0$, but $f(x) = 0$ has no real solutions; so II is not sufficient.

$f(u)f(v) < 0 \Rightarrow$ one of u & v is positive and the other is negative. So the graph of $y = f(x)$ crosses the x-axis, and III is therefore sufficient.

Answer: C

Q10

We need only consider prime values of n (otherwise (*) isn't applicable).

For $n = 2,\ u_2 = 21$ is a multiple of 3, and so we don't have a counter-example.

For $n = 3,\ u_3 = 30$ is a multiple of 3, and so we don't have a counter-example.

For $n = 5,\ u_5 = 44$ is neither a multiple of 3 nor a multiple of 5, and so we have a counter-example.

Answer: E

Q11

[The converse of $x \Rightarrow y$ is $y \Rightarrow x$.]

The attempt effectively proves that if there exist real numbers x & y such that $a = xy(x+y)$ and $b = xy$, then $a^2 - 4b^3 \geq 0$; ie the converse of the problem.

Answer: C

Q12

I: The integrals can be considered to be areas under curves. So if $f(x) \geq g(x)$ in the interval of integration (from 0 up to a non-negative value), then the area represented by $\int_0^x f(t)\, dt$ will be greater than or equal to $\int_0^x g(t)\, dt$; ie I is true.

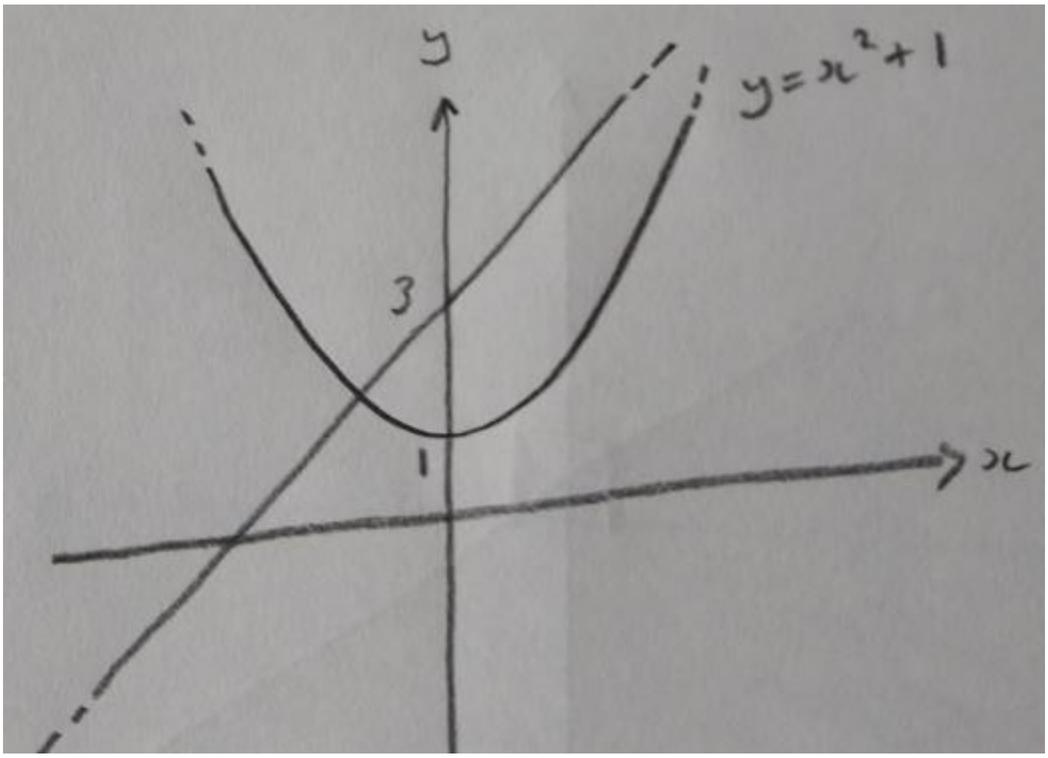
II: Not true; eg when $g(x) = 0$ and $f(x) = (x-1)^2$, the gradient of $f(x)$ is negative for $x < 1$ (being x^2 translated 1 to the right), whilst the gradient of $g(x)$ is always 0.

III: Not true; eg when $g(x) = 1$ and $f(x) = \ln(x+1)$ (being lnx translated 1 to the left): $f'(x) > 0$ for $x \geq 0$, and $g'(x) = 0$. Thus

$f'(x) \geq g'(x)$, but $f(x) < g(x)$ for $x + 1 < e$; ie $x < e - 1$

Answer: B

Q13



From the diagram, I and III are clearly not true.

For II, if $y - x^2 > 0$, then

$y - x < 3 \ \Rightarrow (y - x)(y - x^2) < 3(y - x^2) < 3(1)$

but the argument doesn't hold if $y - x^2 \geq 0$

To find a possible counter-example, consider a case where

$y - x^2 < 0$ [$y - x^2 = 0$ just leads to $0 < 3$]

eg where $y - x^2 = -4$; say $x = 2, y = 0$;

then $(y - x)(y - x^2) = (-2)(-4) = 8$, which is not less than 3

So II is not true either.

Answer: A

Q14

Write $X = 2^x$ (where $X > 0$) and $Y = log_2 y$ (where Y can take any value).

Then the equations become

$pX + Y = 2$ & $X + Y = 1$

Eliminating Y, $pX + (1 - X) = 2$,

so that $X(p - 1) = 1$ & $X = \frac{1}{p-1}$

As $X > 0$, it follows that $p > 1$.

Answer: C

Q15

A circle will be tangential to the y-axis when its equation has the following form:

$(x - p)^2 + (y - q)^2 = p^2$, where p can take any non-zero value (as the radius of the circle is p) and q can take any value.

This expands to give $x^2 - 2px + y^2 - 2qy + q^2 = 0$ (*)

For the circle $x^2 + ax + y^2 + by + c = 0$ (**) to be of this form, we require that

$a = -2p, b = -2q$ & $c = q^2$ for some $p \neq 0$ & some q (***)

Then, as $c = q^2 = \frac{b^2}{4}$, it follows that $b^2 = 4c$.

So this is a necessary condition for the circle to be tangential to the y-axis.

To show that it is a sufficient condition:

If $b^2 = 4c$, then let $p = -\frac{a}{2}$ $(\neq 0)$ & $q = -\frac{b}{2}$.

Then $a = -2p, b = -2q$ & $c = \frac{b^2}{4} = \frac{(-2q)^2}{4} = q^2$, so that the conditions of (***) are met.

Answer: B

Q16

Case 1: $x \geq 0$: $x|x| = px + q \Rightarrow x^2 - px - q = 0$

$x = \frac{p \pm \sqrt{p^2+4q}}{2}$

Case 2: $x < 0$: $x|x| = px + q \Rightarrow x^2 + px + q = 0$

$x = \frac{-p \pm \sqrt{p^2-4q}}{2}$

Conditioning on the sign of q:

If $\boldsymbol{q = 0}$, then for $x \geq 0, x = p \; or \; 0$

And for $x < 0, x = -p \; or \; 0; ie \; x = -p$

(a) If $p = 0$, there is 1 sol'n $(x = 0)$;

(b) If $p > 0$, there are 3 sol'ns $(p, 0 \; \& -p)$,

(c) If $p < 0$, there is 1 sol'n $((x = 0)$

So for $\boldsymbol{q = 0}$, there can be 1 or 3 sol'ns.

If $\boldsymbol{q > 0}$, then there is 1 positive sol'n

For $x < 0$, if $p^2 < 4q$, there are no sol'ns

If $p^2 \geq 4q$ & $p > 0$, then there could be 1 or 2 sol'ns

If $p^2 \geq 4q$ & $p < 0$, then there are no sol'ns

(As $q > 0, p = 0$ is not possible with $p^2 \geq 4q$)

So for $\boldsymbol{q} > \mathbf{0}$, there can be 1, 2 or 3 sol'ns.

If $\boldsymbol{q} < \mathbf{0}$, then:

(a) If $p = 0$, then are no sol'ns for $x \geq 0$

For $x < 0$, there is 1 sol'n.

(b) If $p > 0$, then for $x \geq 0$, $x = \frac{p \pm \sqrt{p^2+4q}}{2}$

and there are 2 sol'ns

For $x < 0$, $x = \frac{-p \pm \sqrt{p^2-4q}}{2}$

and there will be 1 sol'n

(c) If $p < 0$, then for $x \geq 0$, $x = \frac{p \pm \sqrt{p^2+4q}}{2}$

and there are no sol'ns

For $x < 0$, $x = \frac{-p \pm \sqrt{p^2-4q}}{2}$

and there will be 1 sol'n

So for $\boldsymbol{q} < \mathbf{0}$, there can be 1 or 3 sol'ns.

Overall there can be 1, 2 or 3 sol'ns.

Answer: E

Q17

$f(x) = log_2\left(\frac{1}{2}log_2 x\right) = log_2\left(\frac{1}{2}\right) + log_2(log_2 x)$

$g(x) = \frac{1}{2}log_2(log_2 x)$

So $f(x) = -1 + 2g(x)$ (*)

Considering the graphs of $y = -1 + 2u$ and $y = u$

(where $f(x)$ has been replaced by y in (*), and $g(x)$ has been

replaced by u), these intersect when $-1 + 2u = u; ie\ u = 1$,

and $-1 + 2u > u$ for $u > 1$, whilst $-1 + 2u < u$ for $u < 1$

Hence, either $f(x) < g(x) < 1$,

or $f(x) = g(x) = 1$

or $f(x) > g(x) > 1$

Answer: F

Q18

[The question is slightly ambiguous: does "for some choice of x & y" mean (a) "for whatever x & y that are chosen for us", or (b) "there exists some combination of x & y such that ..."
The 1st interpretation is the only one that seems to makes sense.]

Let α & β be acute angles such that $sin\alpha = x$ & $sin\beta = y$.

Then angle A could be α or $180 - \alpha$, and similarly for B.

However, A and B cannot both be obtuse (as the total of the angles in ABC would then exceed 180).

Also, as $x < y, \alpha < \beta$.

We can consider the various possibilities for the angles:

A	B	C	
α	β	$180 - \alpha - \beta$	
$180 - \alpha$	β	$180 - (180 - \alpha + \beta)$ $= \alpha - \beta$	Not possible, as $\alpha < \beta$
α	$180 - \beta$	$180 - (\alpha + 180 - \beta)$ $= \beta - \alpha$	

Thus there are two possibilities for the angles. So, with $AB = 1$ there are just two different triangles (counting congruent triangles as the same).

Answer: C

Q19

$sin\theta\sqrt{1 + sin\theta}\sqrt{1 - sin\theta} + cos\theta\sqrt{1 + cos\theta}\sqrt{1 - cos\theta} = 0$

$\Rightarrow sin\theta\sqrt{1 - sin^2\theta} + cos\theta\sqrt{1 - cos^2\theta} = 0$

Then when $1 \leq \theta \leq 180$, $sin\theta cos\theta + cos\theta sin\theta = 0$

$\Rightarrow sin(2\theta) = 0$

$\Rightarrow 2\theta = 180 \ or \ 360$; ie $\theta = 90 \ or \ 180$

And when $180 < \theta \leq 360$, $sin\theta(-cos\theta) + cos\theta sin\theta = 0$

[noting that $\sqrt{1 - sin^2\theta}$ is by definition positive] and this is satisfied by all values of θ in the range, so that there are 180 solutions

In total there are therefore 182 solutions.

Answer: F

Q20

If $x \geq 0, |x| = x,$ and so $f_n(x) = nx$

If $x < 0, f_1(x) = -x, f_2(x) = |-x + x| = 0,$

$f_3(x) = |0 + x| = -x, f_4(x) = |-x + x| = 0,$ and so on

So $f_{99}(x) = -x$ (when $x < 0$).

Hence $\int_{-1}^{1} f_{99}(x)\, dx = \int_{-1}^{0} -x\, dx + \int_{0}^{1} 99x\, dx$

$= \left[-\frac{1}{2}x^2\right]_{-1}^{0} + \left[\frac{99}{2}x^2\right]_{0}^{1}$

$= -\left(-\frac{1}{2}\right) + \frac{99}{2} = 50$

Answer: E