TMUA 2020 Paper 2 Solutions (11 pages; 30/11/21)

Q1

$x - 2 = x^2 + kx + 2 \Leftrightarrow x^2 + (k-1)x + 4 = 0$

Points of intersection occur when the discriminant is non-negative; ie when $(k-1)^2 - 16 \geq 0$

$\Leftrightarrow (k-1)^2 \geq 16$

$\Leftrightarrow k - 1 \leq -4 \; or \; k - 1 \geq 4$

$\Leftrightarrow k \leq -3 \; or \; k \geq 5$

Answer: E

Q2

First of all, consider $tan\theta = 2$ when θ is in the 1st quadrant.

Then, from a right-angled triangle with adjacent side 1 and opposite side 2, $cos\theta = \frac{1}{\sqrt{5}}$.

The graph of $tan\theta$ has a period of 180°, and $tan\theta = 2$ again when θ is in the 3rd quadrant.

Then $cos(\theta + 180°) = cos\theta cos180° - sin\theta sin180°$

$= \frac{1}{\sqrt{5}}(-1) - 0 = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$

Answer: F

Q3

There is a sign error in line (II). It should read:

$2(9n + 1 - 3n + 1)$

Answer: C

Q4

I is not a counter-example (5 is not greater than 6).

II and III are both counter-examples, as neither can be written in the required form.

Answer: G

Q5

$x \to \infty \Rightarrow y \to 1$, so only A or F could be correct.

Also, $y < 1$, so that only A can be correct.

Answer: A

Q6

The two sides of the equation are areas under the curve $y = f(x)$.

There is no need for there to be symmetry about the y-axis, and so Condition I is not necessary.

Clearly there is no need for $f(x)$ to be constant, and so Condition II is not necessary.

It is clearly possible for $f(x)$ to be positive for $-5 \leq x \leq 5$, and then $f(x) \neq -f(-x)$, so that Condition III is not necessary.

Answer: A

Q7

If $PQ = QR$, $PQRS$ could be a rhombus, so Condition I is not sufficient.

If the diagonals intersect at right-angles, $PQRS$ could be a rhombus (the diagonals split it into 4 congruent right-angle triangles). So Condition II is not sufficient.

If $\angle PQR = \angle QRS$, then $PQRS$ could be a rectangle. So Condition III is not sufficient.

Answer: H

Q8

The proof is fully correct.

Answer: E

Q9

$f(4) = 4sin\left(4 \times \frac{180}{\pi}°\right)$

Answer: F

Q10

$0 < a + b < c + d$ (1) *and* $0 < a + c < b + d$ (2)

(eg $0 < 2 + 4 < 3 + 5$ *and* $0 < 2 + 3 < 4 + 5$)

Inequality I:

(Consider analogy of tennis doubles pairings: d beats a, whether paired with b *or* c.)

Result to prove: $d - a > 0$

From (1), $d - a > b - c$

From (2), $d - a > c - b$

Either $b = c$, in which case $d - a > 0$,

or $b \neq c$, and one of $b - c$ & $c - b$ is > 0, so $d - a > 0$

So (I) is true.

Inequality II: False, from the example given at the start (and the tennis analogy).

Inequality III:

From (1), $(a + b) + (c + d) > 0 + (a + b) > 0$

So (III) is true.

Answer: F

Q11

The pattern for x & $y = \pm 3$ *and* ± 4 is repeated for ± 99 *and* ± 100. As $(-4,3)$ isn't on the spiral, $(-100,99)$ isn't either.

Answer: G

Q12

For the trapezium rule to produce an overestimate, the curve must lie predominantly below the sloping edge of the trapezium.

A, B & C: $f''(x) < 0$ means that the gradient is decreasing, and so the curve will be as in Diagram 1, where there is an underestimate. So A, B & C can be ruled out.

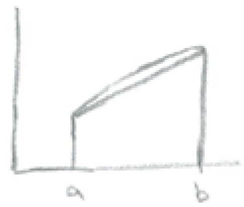

Diagram 1

D: The gradient is increasing, and the curve will be as in Diagram 2, where there is an overestimate. So the answer is D.

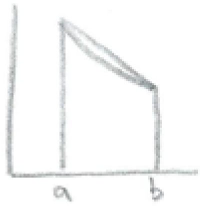

Diagram 2

[E and F are not correct, as the condition "$f'(x) < 0$ & $f''(x) > 0$" is sufficient for there to be an overestimate, but not necessary ("A is true only if B is true" means that $A \Rightarrow B$; ie that B is a necessary condition for A)]

Answer: D

Q13

$\int_0^3 (f(x))^2\, dx + \int_0^3 f(x)\, dx = \int_0^1 f(x)\, dx$

$\Rightarrow \int_0^3 (f(x))^2\, dx = -\int_1^3 f(x)\, dx \quad (*)$

Then, as $\int_0^3 (f(x))^2\, dx \geq 0 \ \left(as \left(f(x)\right)^2 \geq 0\right)$, it cannot be the case that $f(x) > 0$ for all x with $1 \leq x \leq 3$ (otherwise $-\int_1^3 f(x)\,dx < 0$).

So I is necessarily true.

II is equivalent to $\int_0^3 f(x)\,dx - \int_0^1 f(x)\,dx \leq 0$

LHS $= \int_1^3 f(x)\,dx = -\int_0^3 (f(x))^2\,dx$ (from (*)) ≤ 0

So II is necessarily true.

Answer: D

Q14

Property P just means that the last m terms of the sequence sum to zero.

Consider the AP $4,3,2,1,0,-1$, where $ad = 4(-1) < 0$, but property P doesn't hold. So I is not true.

[To see whether a counter-example can be found for II:]

Let the last 2 terms of the sequence be $A, A+d$, such that

$A + (A+d) = 0 \Rightarrow d = -2A$; ie d is even

Instead, let the last 3 terms of the sequence be

$A, A+d, A+2d$, such that $A + (A+d) + (A+2d) = 0$

$\Rightarrow d = -A$

So let $A = 1$, giving a sequence of $4,3,2,1,0,-1$

So d need not be even, and II is not true.

Answer: A

Q15

The terms of the sequence are

$\frac{\sqrt{3}}{2}$, $\frac{\sqrt{3}}{2}$, 0 , $-\frac{\sqrt{3}}{2}$, $-\frac{\sqrt{3}}{2}$, 0 , and then repeating itself.

The sum will be $\frac{\sqrt{3}}{2}$ when $n = 1, 4, 7, 10$, ...

ie when n is one more than a multiple of 3

Answer: D

Q16

The first error occurs on line (III):

In order to use the FTC , we would need to write

$\frac{d}{d(2x)}\left(\int_0^{2x} t^2\, dt\right) = (2x)^2$

[and then $\frac{d}{dx}\left(\int_0^{2x} t^2\, dt\right) = (2x)^2 . \frac{d(2x)}{dx} = 8x^2$]

Answer: D

Q17

Let the two sets be $a\ \ 8\ \ b$ and $c\ \ 9\ \ d$

(where $a < 8 < b$ and $c < 9 < d$)

Then $\frac{a+8+b}{3} = 10$ & $\frac{c+9+d}{3} = 12$,

so that $a + b = 22$ & $c + d = 27$

The smallest possible value for b is 15 (when $a = 7$), and the smallest possible value for d is 20 (when $c = 7$).

With $d = 20$ & $c = 7$, the smallest possible value for a is 6,

and the range is then $20 - 6 = 14$.

(If instead $b = 20$, then $a = 2$, so that the range would be at least $20 - 2 = 18$, and a larger value for b would result in a smaller value for a (possibly negative), but the range would then be larger still.)

Answer: E

Q18

$a = p$ is possible (with $bx^2 + cx + d > qx^2 + rx + s$;
eg with $q = b, r = c, d = s + 1$), so I need not be true

For II: if $b = q$, then $f(x) - g(x) > 0$ for all x

$\Leftrightarrow (a - p)x^3 + (c - r)x + (d - s) > 0$ for all x

No true cubic lies above the x-axis for all x, which means that $a = p$. Then the straight line $y = (c - r)x + (d - s)$ can only lie above the x-axis for all x if its gradient is zero; ie if $c = r$

So II has to be true.

$f(0) - g(0) = d - s$, so that $f(x) - g(x) > 0$ means that $d > s$, and so III has to be true

Answer: G

Q19

All the neighbours of Ts must be L. [A]

Also, there must be at least one T amongst the neighbours of an L. [B]

Suppose that the centre square is T. Then, by [A] and [B], there is only one solution:

TLT
LTL
TLT

So it is possible for $T = 5$ (ie 5 people are telling the truth)

Now suppose that the centre square is L. As each T has to be next to an L, there cannot be more than 4 Ts amongst the 8 outer squares.

So $T \leq 5$.

To establish the minimum value for T:

With an L in the centre, at least one of the squares neighbouring the centre has to be T (by [B]). Suppose, without loss of generality, that it is the top (middle) square. [The grid could be rotated if necessary.] This forces the following:

LTL
?L?
???

Consider the 2 possible cases:

(1)
LTL
?L?
T??

(2)
LTL
?L?
L??

By [A] & [B], these force:

(1)
LTL
LL?
TL?

(2)
LTL
TL?
L??

For (1), the following is possible:
LTL
LLL
TLT

but the two missing squares cannot both be L (by [B]).

So, for (1), $T > 2$

For (2), the missing squares cannnot all be L (by [B]).

So, for (2) also, $T > 2$

Hence $3 \leq T \leq 5$.

Answer: E

Q20

Not F, as otherwise B would be true as well.

A ($x \geq 0$ only if $f(x) < 0$) $\equiv x \geq 0 \Rightarrow f(x) < 0$

and B ($x < 0$ if $f(x) \geq 0$) $\equiv f(x) \geq 0 \Rightarrow x < 0$

A and B are contrapositives of each other,

and therefore $A \equiv B$, so not A or B (as if A is true, then B will be true, and vice-versa)

Similarly, $D \equiv E$, so not D or E

Answer: C