TMUA 2020 Paper 1 Solutions (13 pages; 16/11/21)

Q1

If $f(x) = \frac{x^3-5x^2}{2x\sqrt{x}} = \frac{1}{2}x^{\frac{3}{2}} - \frac{5}{2}x^{\frac{1}{2}}$,

$f'(x) = \frac{3}{4}x^{\frac{1}{2}} - \frac{5}{4}x^{-\frac{1}{2}} = \frac{3x-5}{4\sqrt{x}}$

Answer: C

Q2

$f(x) = 2x^3 + px^2 + q$

$f\left(-\frac{1}{2}\right) = 0 \Rightarrow -\frac{2}{8} + p\left(\frac{1}{4}\right) + q = 0 \Rightarrow p + 4q = 1$ (1)

$f(2) = 0 \Rightarrow 2(8) + p(4) + q = 0 \Rightarrow 4p + q = -16$ (2)

and from (1), $4p + 16q = 4$ (3)

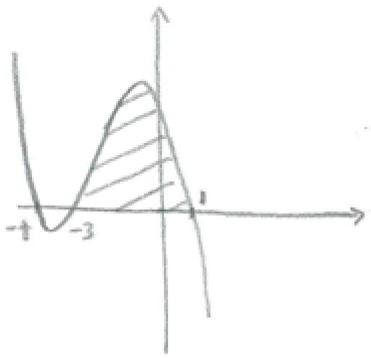
Then (3) − (2) $\Rightarrow 15q = 20 \Rightarrow q = \frac{4}{3}$ & $p = 1 - 4\left(\frac{4}{3}\right) = -\frac{13}{3}$,

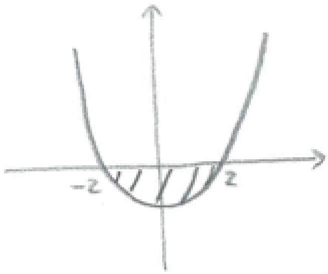
so that $2p + q = -\frac{26}{3} + \frac{4}{3} = -\frac{22}{3}$

Answer: C

Q3

$(x+4)(x+3)(1-x) > 0$ & $(x+2)(x-2) < 0$





Sol'n is $-2 < x < 1$

Answer: B

Q4

Let the terms of the GP be a, ar & ar^2, and the terms of the AP be $a, a + 3d$ & $a + 5d$

Then $\frac{a}{1-r} = 12$ (1), $ar = a + 3d$ (2) & $ar^2 = a + 5d$ (3)

(2) $\Rightarrow 5ar = 5a + 15d$ (4) & (3) $\Rightarrow 3ar^2 = 3a + 15d$ (5)

Then (4) – (5) $\Rightarrow 5ar - 3ar^2 = 2a$

$\Rightarrow 3r^2 - 5r + 2 = 0$

$\Rightarrow (3r - 2)(r - 1) = 0$

$\Rightarrow r = 1$ (reject, as GP has sum to infinity) $or\ r = \frac{2}{3}$

Then, from (1), $\frac{a}{1-\frac{2}{3}} = 12 \Rightarrow a = 4$

Answer: D

Q5

$y = px^2 + 6x - q = p(x + \frac{1}{4})^2$

$\Rightarrow 6 = \frac{p}{2}$ and $-q = \frac{p}{16}$,

so that $p = 12$ and $q = -\frac{3}{4}$, and hence $p + 8q = 6$

Answer: A

Q6

Write $y = 5^x$

We need to minimise $y^2 - 4y + 7$

This occurs when $y = \frac{-b}{2a}$ in the quadratic formula;

ie when $y = 2$

and $f(x) = \frac{1}{2^2 - 4(2) + 7} = \frac{1}{3}$

Answer: C

Q7

$2^{3x} = 8^{y+3}$ & $4^{x+1} = \frac{16^{y+1}}{8^{y+3}}$

Let $a = 2^x$ & $b = 2^y$

Then $a^3 = 8^3 \times b^3$ & $4a^2 = \frac{16b^4}{8^3 \times b^3} = \frac{b}{32}$

$\Rightarrow a = 8b$ & $b = 128a^2$,

so that $b = 128(64b^2) \Rightarrow b = \frac{1}{2^{13}}$

So $2^y = b = 2^{-13} \Rightarrow y = -13$

and $2^x = a = 8b = 8 \times 2^{-13} \Rightarrow 2^x = 2^{-10} \Rightarrow x = -10$

Hence $x + y = -10 - 13 = -23$

Answer: A

Q8

$(p - x)(x + 2) < 4\ \forall x \ \Leftrightarrow x^2 + (2 - p)x + 4 - 2p > 0\ \ \forall x$

$\Leftrightarrow \Delta < 0$

ie $(2 - p)^2 - 4(4 - 2p) < 0$

$\Leftrightarrow (2 - p)(2 - p - 8) < 0$

$\Leftrightarrow (2 - p)(p + 6) > 0$

$\Leftrightarrow\ -6 < p < 2$

Answer: D

Q9

[The usual method of obtaining a polynomial with roots related to those of a given polynomial (namely via the substitution $u = \sqrt{x}$) doesn't work, as it produces a quartic (containing the spurious roots $u = -\sqrt{x}$]

Let the required quadratic be $x^2 + bx + c$

Then $-b = \sqrt{\alpha} + \sqrt{\beta}$ and $c = \sqrt{\alpha}\,.\sqrt{\beta}$

From the original eq'n, $\alpha + \beta = 14$ and $\alpha\beta = 9$

So $c = \sqrt{9} = 3$, and $b^2 = (\sqrt{\alpha} + \sqrt{\beta})^2 = \alpha + \beta + 2\sqrt{\alpha}\,.\sqrt{\beta}$

$= 14 + 2(3) = 20$

And, as $-b = \sqrt{\alpha} + \sqrt{\beta}$, $b < 0$, so that $b = -\sqrt{20}$

So the required quadratic is $x^2 - \sqrt{20}x + 3$

Answer: C

Q10

Translation by $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$: $y = 4x^2 \rightarrow y = 4(x-3)^2 - 5$

Reflection in the x-axis:

$y = 4(x-3)^2 - 5 \rightarrow y = -[4(x-3)^2 - 5]$

Stretch parallel to the x-axis with scale factor 2:

$y = -[4(x-3)^2 - 5] \rightarrow y = -[4\left(\frac{x}{2} - 3\right)^2 - 5]$

$= -x^2 + 12x - 31$

Answer: A

Q11

$R = S \Rightarrow \int_0^2 a(x-2)(x-q)\,dx = -\int_2^q a(x-2)(x-q)\,dx$

$\Rightarrow \int_0^2 x^2 - (2+q)x + 2q\,dx = -\int_2^q x^2 - (2+q)x + 2q\,dx$

$\Rightarrow \left[\frac{1}{3}x^3 - \frac{1}{2}(2+q)x^2 + 2qx\right]_0^2$

$= -\left[\frac{1}{3}x^3 - \frac{1}{2}(2+q)x^2 + 2qx\right]_2^q$

$\Rightarrow \frac{1}{3}q^3 - \frac{1}{2}(2+q)q^2 + 2q^2 = 0$

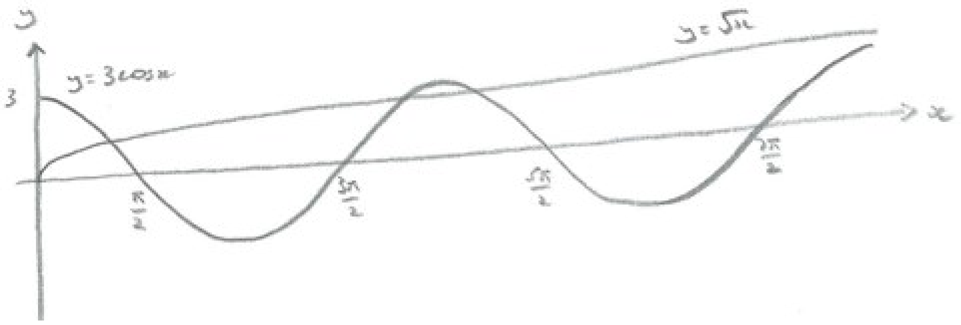
$\Rightarrow \frac{q^2}{6}(2q - 3(2+q) + 12) = 0$

$\Rightarrow -q + 6 = 0$ (as $q > 2$)

$\Rightarrow q = 6$

Answer: E

Q12



Considering the graphs of $y = 3cosx$ and $y = \sqrt{x}$, there will be one intersection point where $0 < x < \frac{\pi}{2}$.

When $x = 2\pi$, $3cosx = 3$ and $\sqrt{x} = \sqrt{2\pi} < \sqrt{2(4)} < \sqrt{9} = 3$;

ie $\sqrt{x} < 3cosx$

So there will be another two points of intersection where

$\frac{3\pi}{2} < x < \frac{5\pi}{2}$

When $x = \frac{3\pi}{2} + 2\pi$, $\sqrt{x} = \sqrt{\frac{7\pi}{2}} > \sqrt{\frac{7(3)}{2}} > \sqrt{10} > \sqrt{9} = 3$

So there will be no further points of intersection.

Thus there are 3 points of intersection in total.

Answer: D

Q13

$(1+x+y^2)^7 = (1+x+y^2)(1+x+y^2)\ldots(1+x+y^2)$

x^2y^4 could arise from eg $x, x, y^2, y^2, 1, 1, 1$

Number of ways $= \frac{7!}{2!2!3!} = \frac{7(6)(5)(4)}{4} = 210$

Answer: F

Q14

By symmetry, $2\int_0^a mx - x^3\, dx = 6\ (m > 0)$

where $y = mx$ and $y = x^3$ intersect when $x = a > 0$,

so that $ma = a^3 \Rightarrow a = \sqrt{m}$ (as $a \neq 0$ & $a > 0$)

So $\left[\frac{m}{2}x^2 - \frac{1}{4}x^4\right]_0^{\sqrt{m}} = 3$

$\Rightarrow 2m^2 - m^2 = 12$

$\Rightarrow m = \sqrt{12} = 2\sqrt{3}$

Answer: E

Q15

$(log_2 x)^4 + 12(log_2(\frac{1}{x}))^2 - 2^6 = 0$

Let $y = log_2 x$, so that $y^4 + 12y^2 - 64 = 0$

$\Rightarrow (y^2+16)(y^2-4) = 0$

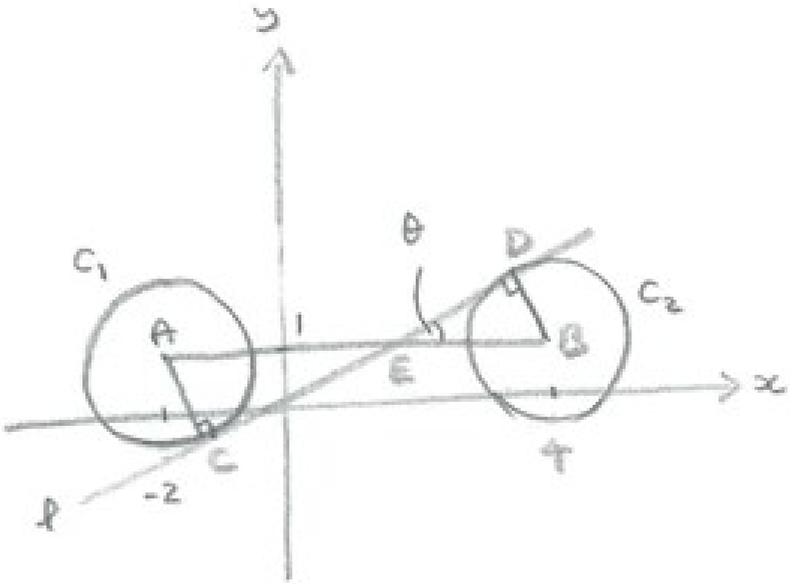
$\Rightarrow y^2 = 4 \Rightarrow y = \pm 2$

Then $y = 2 \Rightarrow x = 4$, and $y = -2 \Rightarrow x = \frac{1}{4}$

and the positive difference between these values is $4 - \frac{1}{4} = \frac{15}{4}$

Answer: C

Q16



The triangles ACE and BDE are similar (each having a right angle, and sharing the angle at E; ie θ).

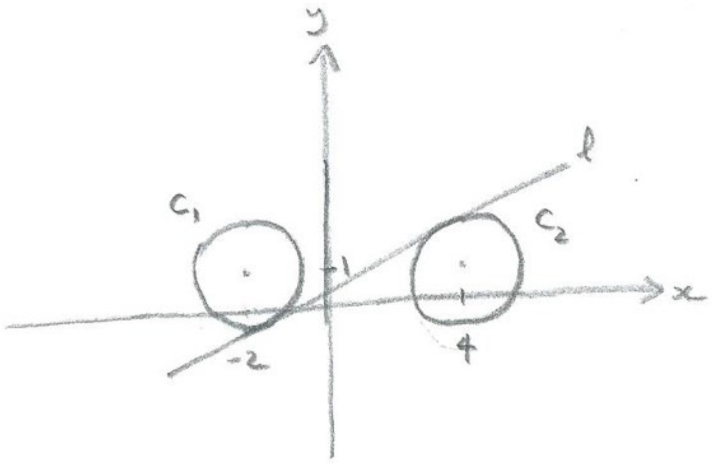
Also, the sides opposite θ (AC and BD) are both equal, and so the triangles are in fact congruent (this also follows by symmetry, as the circles are of the same size – the diagram is not quite to scale).

As $AB = 6, sin\theta = \frac{AC}{AE} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$

Then $tan\theta = \frac{1}{\sqrt{3-1}} = \frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$

Answer: C

Alternative Method (too time-consuming for the exam)



Let l have eq'n $y = mx + c$ [we need to find $m = tan\theta$]

As l is a tangent to both circles, there are repeated roots for the following eq'ns:

$(x + 2)^2 + (mx + c - 1)^2 = 3$ (1)

and $(x - 4)^2 + (mx + c - 1)^2 = 3$ (2)

and so the discriminants of (1) & (2) (as quadratics in x) must both be zero.

Writing $d = c - 1$,

(1) becomes $x^2(1 + m^2) + x(4 + 2md) + 4 + d^2 - 3 = 0$

Then $\Delta = 0 \Rightarrow (4 + 2md)^2 - 4(1 + m^2)(1 + d^2) = 0$

[Fortunately, we can see that this contains $4m^2d^2 - 4m^2d^2$]

$\Rightarrow (2 + md)^2 - (1 + m^2)(1 + d^2) = 0$

$\Rightarrow m^2(-1) + m(4d) + 4 - (1 + d^2) = 0$

$\Rightarrow m^2 - 4dm + d^2 - 3 = 0$ (3)

And (2) becomes $x^2(1+m^2)+x(-8+2md)+16+d^2-3=0$

Then $\Delta=0 \Rightarrow (-8+2md)^2-4(1+m^2)(13+d^2)=0$

$\Rightarrow (-4+md)^2-(1+m^2)(13+d^2)=0$

$\Rightarrow m^2(-13)+m(-8d)+16-(13+d^2)=0$

$\Rightarrow 13m^2+8dm+d^2-3=0$ (4)

Then, from (3) & (4), $m^2-4dm=13m^2+8dm$

[this approach only works because it reduces to a linear eq'n in m]

$\Rightarrow 12m^2+12dm=0$

$\Rightarrow m+d=0$, as $m\neq 0$ (we are told that the gradient is positive)

Then (3) $\Rightarrow m^2-4(-m)m+(-m)^2-3=0$

$\Rightarrow 6m^2=3$, so that $m=\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$ (as $m>0$)

Q17

$mx+c=\sqrt{x} \Rightarrow m^2x^2+2mcx+c^2=x$

[but beware of spurious sol'n, resulting from $mx+c=-\sqrt{x}$]

$\Rightarrow m^2x^2+(2mc-1)x+c^2=0$

2 points of intersection $\Leftrightarrow$ discriminant >0

ie $(2mc-1)^2-4m^2c^2>0$

$\Leftrightarrow -4mc+1>0$

$\Leftrightarrow m<\frac{1}{4c}$

This is a necessary (but possibly not sufficient) condition for 2 points of intersection.

Considering the graphs of $y = mx + c \;\; and \;\; y = \sqrt{x}$, there will only be 2 points of intersection if $m > 0$.

So $0 < m < \frac{1}{4c}$ is also a necessary condition for 2 points of intersection.

This means that A is the only possible answer (though we haven't proved that $0 < m < \frac{1}{4c}$ is a sufficient condition).

Answer: A

Q18

For $0 \leq x \leq 90, 1 - 2cos^2x = cosx \Leftrightarrow 2cos^2x + cosx - 1 = 0$

$\Leftrightarrow (2cosx - 1)(cosx + 1) = 0$

Rejecting $cosx = -1$ (as $0 \leq x \leq 90$), $cosx = \frac{1}{2}$, so that $x = 60°$

For $90 < x \leq 180, 1 - 2cos^2x = -cosx$

$\Leftrightarrow 2cos^2x - cosx - 1 = 0$

$\Leftrightarrow (2cosx + 1)(cosx - 1) = 0$

Rejecting $cosx = 1$ (as $90 < x \leq 180$), $cosx = -\frac{1}{2}$,

so that $x = 120°$

Thus sol'ns are $x = 60°$ & $120°$, and the sum is $180°$.

Answer: A

Q19

Consider the graph of $y = x^2 - 52x - 52$

The positive root of $x^2 - 52x - 52 = 0$ is

$\alpha = \frac{52+\sqrt{52^2+4(52)}}{2}$ (noting that $\frac{52-\sqrt{52^2+4(52)}}{2} < 0$)

$= 26 + \sqrt{52(13+1)} = 26 + 2\sqrt{13(14)}$

$< 26 + 2(14) = 54$

Also $26 + 2\sqrt{13(14)} > 26 + 2(13) = 52$

So $52 < \alpha < 54$, and therefore the answer could be 53 or 54.

Consider $\alpha = 26 + 2\sqrt{13(14)} < 53$

$\Leftrightarrow 2\sqrt{13(14)} < 27$

$\Leftrightarrow 4(13)(14) < 27^2 = 81(9) = 729$

$\Leftrightarrow 52(14) < 729$

$\Leftrightarrow 520 + 208 < 729$

$\Leftrightarrow 728 < 729$

So $\alpha < 53$ and hence $52 < \alpha < 53$, making the answer 53

Answer: E

Q20

[Taking a to be a real number]

Either (i) $x^2 - x + a = (x-a)(x-b)$, where $b \neq a$

or (ii) $x^2 - x + a = (x-b)^2$, again where $b \neq a$

(i)$\Rightarrow a + b = 1$ & $ab = a \Rightarrow a = 0$ & $b = 1$

(ii)$\Rightarrow 2b = 1$ & $b^2 = a \Rightarrow b = \frac{1}{2}$ & $a = \frac{1}{4}$

So there are two possible values of a.

Answer: C