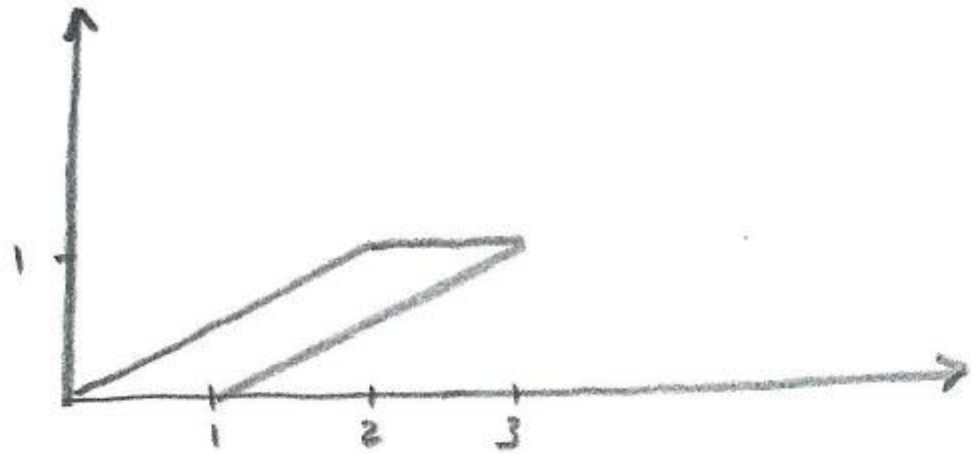
Shears (8 pages; 12/8/18)

(1) A simple example of a shear is shown below. This is the image of the unit square. As the area of the parallelogram (perpendicular height × base) is 1, the area scale factor of the transformation is 1, and so the determinant of the matrix will be 1. This is in fact true of all shears (and is derived later on).

The matrix corresponding to this shear will be $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ (based on the images of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$).



(2) A general shear involves the **line of shear**, which is a line of invariant points, and the **shear factor**, which gives the distance moved by a point (in the direction of the line of shear) as a multiple of its perpendicular distance from the line of shear.

For the example in (1), the **line of shear** is the x-axis.

Note that $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$; ie the x-axis is a line of invariant points.

Also, consider $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ b \end{pmatrix} = \begin{pmatrix} 2b \\ b \end{pmatrix}$, which show that the shear factor is 2 (as the point $\begin{pmatrix} 0 \\ b \end{pmatrix}$ has moved $2b$ to the right).

(3) Shears are often described by giving the image of a particular point, as well as the line of shear. This is usually more helpful than stating the shear factor (in addition to the line). The formula for the shear factor, in terms of the elements of the matrix, is derived later on.

(4) Since all shears have a line of invariant points (ie the line of shear), they must have an eigenvalue of 1 associated with them. Also, there will not be another invariant line passing through the Origin, because a point on that line (other than the Origin) would have to transform in the direction of the line of shear, and this would give an image that wasn't on the invariant line. Hence there can only be a single eigenvalue of 1; ie there must be two repeated roots of 1 for the auxiliary equation.

Suppose that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ represents a shear. The auxiliary equation is

$(a - \lambda)(d - \lambda) - bc = 0$

or $\lambda^2 - (a + d)\lambda + ad - bc = 0$

In order for there to be repeated roots of 1,

$a + d = 2$ (ie the sum of the roots) and $ad - bc = 1$ (the product of the roots)

Thus the determinant of the matrix is always 1, and the trace (defined to be the sum of the elements on the leading diagonal) is always 2.

Note that $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, from (1), satisfies the trace requirement.

(5) It can also be shown that if $a + d = 2$ and $ad - bc = 1$, then $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ represents a shear.

Solution

From (4), there will be a repeated eigenvalue of 1. Then an eigenvector of $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ will be given by

$\begin{pmatrix} a-1 & c \\ b & d-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so that $(a-1)x + cy = 0$

[and $bx + (d-1)y = 0$ can be shown to be equivalent]

Thus an eigenvector is $\begin{pmatrix} c \\ 1-a \end{pmatrix}$.

Note then that the equation of the line of shear will be $\boldsymbol{y = \frac{(1-a)x}{c}}$

[or alternatively $y = \frac{bx}{1-d}$]

[Note that $\frac{(\mathbf{1-a})}{\mathbf{c}} = \frac{b}{1-d} \Leftrightarrow (1-a)(1-d) = bc$

$\Leftrightarrow ad - bc - (a+d) + 1 = 0$

and LHS $= 1 - 2 + 1 = 0$]

We want to show that, for a general point $\begin{pmatrix} p \\ q \end{pmatrix}$,

$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} + k\begin{pmatrix} c \\ 1-a \end{pmatrix}$

[ie that the image of a general point is the result of a translation in the direction of the line of shear]

Now, $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} + [\begin{pmatrix} a & c \\ b & d \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}]\begin{pmatrix} p \\ q \end{pmatrix}$

$= \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} a-1 & c \\ b & d-1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} (a-1)p + cq \\ bp + (d-1)q \end{pmatrix}$ (1)

We need to show that $\frac{bp+(d-1)q}{(a-1)p+cq} = \frac{1-a}{c}$ (1a)

Eliminating bc from $ad - bc = 1$, this is equivalent to

$(ad-1)p+(d-1)qc=(1-a)(a-1)p+(1-a)cq$

Then eliminating d from $a+d=2$,

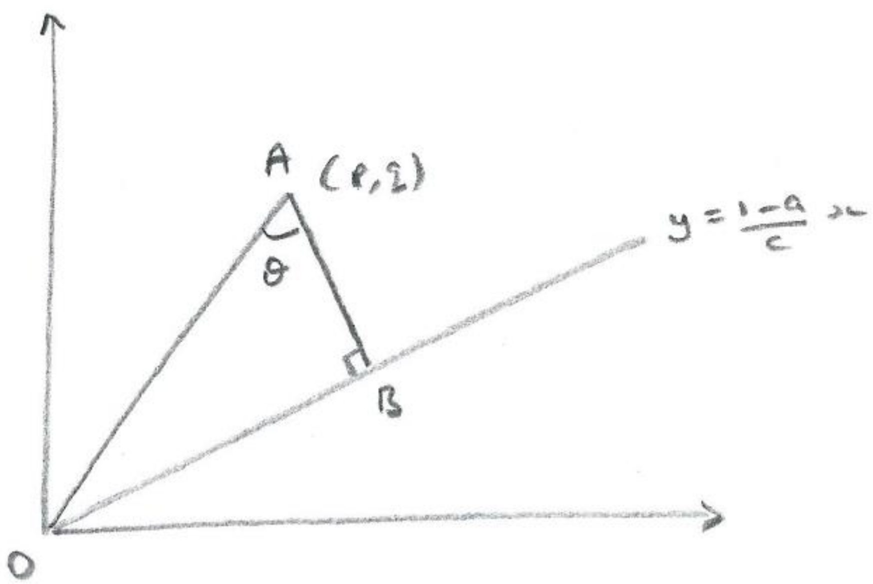
LHS $=(a(2-a)-1)p+(1-a)qc=$RHS

So we have shown that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}=\begin{pmatrix} p \\ q \end{pmatrix}+k\begin{pmatrix} c \\ 1-a \end{pmatrix}$ (2)

Also, from (1) & (2), $kc=(a-1)p+cq$, so that $k=\frac{(a-1)p+cq}{c}$ (3)

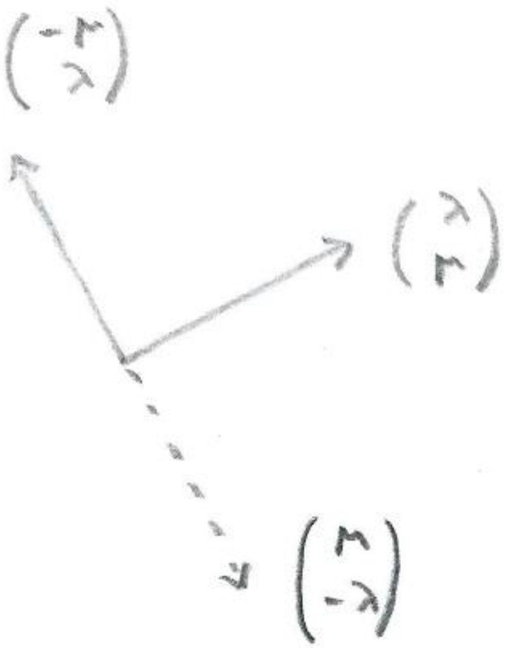
We also need to show that the displacement of the general point is proportional to its perpendicular distance from the line of shear.

Referring to the diagram below, where the point (p,q) is to the left of the direction $\begin{pmatrix} c \\ 1-a \end{pmatrix}$, $\overrightarrow{BA}=\begin{pmatrix} a-1 \\ c \end{pmatrix}$



(in the diagram, $\frac{1-a}{c}>0$, but without loss of generality)

[See the further diagram below for the general case, which can be verified by considering the signs of λ & μ, for different orientations of $\begin{pmatrix}\lambda\\ \mu\end{pmatrix}$.]



Then the perpendicular distance AB is the component of $\overrightarrow{AO}$ in the direction $\overrightarrow{AB}$, which is $\overrightarrow{AO}.\frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \begin{pmatrix}-p\\ -q\end{pmatrix}.\frac{-\begin{pmatrix}a-1\\ c\end{pmatrix}}{\sqrt{(a-1)^2+c^2}}$

Writing $\alpha = \sqrt{(a-1)^2 + c^2}$, $AB = [p(a-1) + qc]/\alpha$

From (2) & (3), the displacement of (p, q) is

$k\begin{pmatrix}c\\ 1-a\end{pmatrix} = \frac{(a-1)p+cq}{c}\begin{pmatrix}c\\ 1-a\end{pmatrix}$;

ie a displacement of $\left(\frac{(a-1)p+cq}{c}\right)\alpha$ in the direction $\begin{pmatrix}c\\ 1-a\end{pmatrix}$

The ratio of this displacement to $\boldsymbol{AB}$ is $\boldsymbol{\frac{\alpha^2}{c} = \frac{(a-1)^2+c^2}{c}}$

(when the point (p, q) is to the left of the line of shear, in the direction $\begin{pmatrix} c \\ 1-a \end{pmatrix}$)

Note that this ratio is constant; ie independent of (p, q).

If (p, q) is to the right of the line of shear, in the direction $\begin{pmatrix} c \\ 1-a \end{pmatrix}$, then

$$AB = \begin{pmatrix} -p \\ -q \end{pmatrix} . \frac{-\begin{pmatrix} 1-a \\ -c \end{pmatrix}}{\sqrt{(a-1)^2+c^2}} = -[p(a-1) + qc]/\alpha$$

As the displacement is still $\left(\frac{(a-1)p+cq}{c}\right)\alpha$ in the direction $\begin{pmatrix} c \\ 1-a \end{pmatrix}$,

the ratio of this displacement to AB is $-\frac{\alpha^2}{c} = -\frac{(a-1)^2+c^2}{c}$

when (p, q) is to the right , instead of left; ie the displacement will be in the opposite direction.

[Note of course that c could be negative - producing a displacement in the opposite direction to $\begin{pmatrix} c \\ 1-a \end{pmatrix}$ when the point is 'to the left'.]

Thus, from (4) & (5), **the necessary and sufficient condition for a shear is that the determinant of the matrix is 1, and the trace is 2.**

[Note: An alternative formula to $\frac{(a-1)^2+c^2}{c}$ can be derived by writing $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} + k'\begin{pmatrix} 1-d \\ b \end{pmatrix}$ (4)

Referring to (1a), we have already established that

$\frac{bp+(d-1)q}{(a-1)p+cq} = \frac{1-a}{c} = \frac{b}{1-d}$, so that (4) is permissible.

Then, from (1) & (4), $k'b = bp + (d-1)q$

and $AB = \begin{pmatrix} -p \\ -q \end{pmatrix} . \frac{-\begin{pmatrix} -b \\ 1-d \end{pmatrix}}{\sqrt{(-b)^2+(1-d)^2}}$ (as before, but with the direction of the line of shear being $\begin{pmatrix} 1-d \\ b \end{pmatrix}$ instead of $\begin{pmatrix} c \\ 1-a \end{pmatrix}$). This assumes that the point is 'to the left'.

Writing $\beta = \sqrt{b^2 + (1-d)^2}$, $AB = [-pb + q(1-d)]/\beta$

The displacement of (p,q) is $k' \begin{pmatrix} 1-d \\ b \end{pmatrix} = \frac{bp+(d-1)q}{b} \begin{pmatrix} 1-d \\ b \end{pmatrix}$;

ie a displacement of $\left(\frac{bp+(d-1)q}{b}\right)\beta$ in the direction $\begin{pmatrix} 1-d \\ b \end{pmatrix}$

The ratio of this displacement to AB is $-\frac{\beta^2}{b} = -\frac{b^2+(1-d)^2}{b}$

(when the point (p,q) is 'to the left'). As before, the sign is reversed when the point is 'to the right'.]

(6) Summary of results for the shear with matrix $\begin{pmatrix} \boldsymbol{a} & \boldsymbol{c} \\ \boldsymbol{b} & \boldsymbol{d} \end{pmatrix}$

(i) $ad - bc = 1$ & $a + d = 2$

(ii) There is a repeated eigenvalue of 1.

(iii) The equation of the line of shear is $y = \frac{(1-a)x}{c}$

(or $y = \frac{bx}{1-d}$)

(iv) The shear factor is $\frac{(a-1)^2+c^2}{c}$ (when the point is to the left of the line of shear, in the direction $\begin{pmatrix} c \\ 1-a \end{pmatrix}$). So that if the perpendicular distance of the point from the line of shear is D, then the displacement of the point under the shear will be $\frac{(a-1)^2+c^2}{c} D$ in the direction of $\begin{pmatrix} c \\ 1-a \end{pmatrix}$.

If the point is to the right, instead of left, then the displacement will be in the opposite direction.

[Alternatively, the shear factor is $-\frac{b^2+(1-d)^2}{b}$ (when the point is 'to the left').]