Series – Q7 [Practice/M] (20/6/21)

Starting from $\sum_{r=1}^{n}(r+1)^2 = \left(\sum_{r=1}^{n} r^2\right) + 2\left(\sum_{r=1}^{n} r\right) + n$,

make the substitution $R = r + 1$,

to prove that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

Solution

Let $R = r + 1$, so that

$\text{LHS} = \sum_{R=2}^{n+1} R^2 = (\sum_{R=1}^{n} R^2) + (n+1)^2 - 1$

Thus $(\sum_{R=1}^{n} R^2) + (n+1)^2 - 1 = (\sum_{r=1}^{n} r^2) + (2\sum_{r=1}^{n} r) + n$,

As $(\sum_{R=1}^{n} R^2)$ and $(\sum_{r=1}^{n} r^2)$ are equal, it follows that

$2\sum_{r=1}^{n} r = (n+1)^2 - 1 - n = n^2 + n = n(n+1)$

and so $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

Note: This method is a variant on the "Method of Differences".