Selections (7 pages; 15/3/24)

Contents

(1) Interpretations of $\binom{n}{r}$

(2) Number of ways of arranging 5 items where order is important

(3) Number of ways of selecting 2 items from 5 where order is important ("Permutations")

(4) Number of ways of selecting 2 items from 5 where order is NOT important ("Combinations")

(5) Number of ways of arranging 3 Ss and 2Fs, where the pattern of Ss and Fs is important

(6) Interpretation of ${}^5C_2$ as the Binomial coefficient in the expansion of $(a+b)^5$

(7) $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$

(8) Classification of selections

(1) Interpretations of $\binom{n}{r}$

(a) Number of ways of selecting r items from n

eg for the items A, B, C, D, E

$\binom{5}{2} = \#(AB, AC, AD, AE, BC, BD, BE, CD, CE, DE) = 10$

[this notation isn't standard, by the way]

(b) Number of ways of choosing r positions out of n

eg for $\binom{5}{2}$, one such way would be $\_X\_X\_$

(2) Number of ways of arranging 5 items where order is important

There are 5 ways of filling the 1st place.

Then, for each of these, there are 4 ways of filling the 2nd place; then for each of these 20 ways, there are 3 ways of filling the 3rd place, and so on.

So the number of ways = 5 x 4 x 3 x 2 x 1 = 5!

This is a special case of (2):

(3) Number of ways of selecting 2 items from 5 where order is important ("Permutations")

eg placing 2 horses in a 5-horse race (P for place ⇒ P for Permutation)

Method 1

There are 5 ways of selecting the 1st item. Then, for each of these 5 ways, there are 4 ways of selecting the 2nd item.

Hence, $^5P_2$ = 5 x 4

Method 2 (more complicated, but helps to understand Combinations - covered below)

From (1), the number of ways of arranging all 5 items is 5!

ABCDE, ABCED, ABDCE, ABDEC, ABECD & ABEDC all count as the same selection, if we are only interested in the first two letters – and there are 3! of these (the number of ways of arranging CDE). Similarly for all other pairs of 2 letters.

Therefore we need to divide by 3! to remove the duplication, to get:

$$^5P_2 = \frac{5!}{3!} = 5 \text{ x } 4$$

This method of removing duplication will be used again below.

(4) Number of ways of selecting 2 items from 5 where order is NOT important ("Combinations")

eg choosing 2 people out of 5 to form a Committee (C for Committee ⇒ C for Combination)

Because order is not important, ABCDE is treated as the same thing as BACDE.

So the $^5P_2$ ways in (2) contain duplication which is removed, as before, by dividing by 2! (the number of ways of arranging 2 letters)

Hence $^5C_2 = \frac{5!}{3!2!}$

(5) Number of ways of arranging 3 Ss and 2Fs, where the pattern of Ss and Fs is important

This can represent the possible arrangements of successes and failures in 5 trials, to give the coefficient in the Binomial probability.

Let the 3 Ss be labelled S_1 , S_2 & S_3 and the 2 Fs, F_1 and F_2.

There are 5! ways of arranging the letters if they are thought of as all different.

However, if the Ss are considered to be indistinguishable, all the following count as the same arrangement:

$S_1F_1S_2S_3F_2$

$S_1F_1S_3S_2F_2$

$S_2F_1S_1S_3F_2$

$S_2F_1S_3S_1F_2$

$S_3F_1S_1S_2F_2$

$S_3F_1S_2S_1F_2$

As before, the duplication is removed by dividing by 3!

If the Fs are also to be treated as indistinguishable, then we do the same thing with them and divide by 2!

So the number of ways of arranging 3 Ss and 2Fs, where the pattern of Ss and Fs is important is: $\frac{5!}{3!2!}$

Although this is a case where order <u>is</u> important (in that the pattern of Ss and Fs is important), it has the same form as $^5C_2$. This can also be explained as follows:

We are interested in the different places that the Fs can occupy.

For example, SSSFF $\Rightarrow$ places 4&5 SSFSF $\Rightarrow$ places 3&5

As 'places 4&5' and 'places 5&4' would count as the same thing, the answer is $^5C_2$ (the number of ways of selecting 2 places from 5, where order doesn't matter).

(6) Interpretation of $^5C_2$ as the Binomial coefficient in the expansion of $(a+b)^5$

$(a+b)^5 = (a+b)(a+b)(a+b)(a+b)(a+b)$

The terms involving a^2 (and b^3) are obtained by selecting 2 of the 5 brackets (these 2 give rise to the "a"s and the other 3 give rise to the "b"s)

This can be done in $^5C_2$ ways [from (3)]; ie the term a^2b^3 occurs $^5C_2$ times.

(7) $\boldsymbol{\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}}$

[where $\binom{n}{r}$ is written instead of ^nC_r]

Proof

If r items are to be chosen from n items, then either the 1st item is included or it isn't.

If it is included, then there are $\binom{n-1}{r-1}$ ways of choosing the remaining $r-1$ items that are required.

If it isn't included, then there are $\binom{n-1}{r}$ ways of choosing the remaining r items that are required.

This gives a total of $\binom{n-1}{r-1} + \binom{n-1}{r}$ ways of choosing the r items.

(8) Classification of selections

(i) Ordered selections with repetition

Number of ways of selecting r items from n, if repetitions are allowed, and order is important $= n^r$

(ii) Ordered selections without repetition

Number of ways of selecting r items from n, if repetitions are not allowed, and order is important

$= n(n-1) \dots (n-[r-1]) = n(n-1) \dots (n-r+1)$

[Known as a Permutation]

$P(n,r)$ or ${}^nP_r = \frac{n!}{(n-r)!} = n(n-1) \dots (n-r+1)$

(iii) Unordered selections without repetition

Number of ways of selecting r items from n, if repetitions are not allowed, and order is not important

[Known as a Combination.]

$C(n,r)$ or ${}^nC_r$ or $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\dots(n-r+1)}{r!}$

[${}^nC_r$ can be obtained from ${}^nP_r = \frac{n!}{(n-r)!}$ by dividing by $r!$, to remove duplication (the ${}^nP_r$ ordered ways can be divided into groups of $r!$, containing the same items, but in a different order).]

(iv) Unordered selections with repetition

Number of ways of selecting r items from n, if repetitions are allowed, and order is not important

eg $BBCE$ selected from $ABCDEF$ $(r = 4, n = 6)$

write as $|XX|X||X|$

(| indicates that we are moving on to the next letter, and XX indicates that we are selecting 2 items from the current letter: so

$|XX|X||X|$ means: move on to B (without selecting any As); then select 2 Bs; then move on to the Cs; select 1 C; move on to D, and then on to E; select 1 E; then move on to F, but select no Fs)

$=$ Number of ways of choosing r positions for the Xs,

out of the $n-1$ $|s$ and r Xs (giving a total of $n-1+r$)

$= \binom{n-1+r}{r}$