Scalar Product (4 pages; 19/9/20)

(1) The scalar (or 'dot') product of the vectors $\underline{a}$ and $\underline{b}$ (which can be 2D or 3D) is defined as:

$\underline{a}.\underline{b} = |\underline{a}||\underline{b}|cos\theta$, where θ is the angle between $\underline{a}$ and $\underline{b}$.

Note that $\underline{b}.\underline{a} = \underline{a}.\underline{b}$ (as $\cos(-\theta) = cos\theta$)

(2) $\underline{i}.\underline{i} = \underline{j}.\underline{j} = \underline{k}.\underline{k} = 1$

and $\underline{i}.\underline{j} = \underline{j}.\underline{k} = \underline{k}.\underline{i} = 0$

So $\left(a_1\underline{i} + a_2\underline{j} + a_3\underline{k}\right).\left(b_1\underline{i} + b_2\underline{j} + b_3\underline{k}\right) = a_1b_1 + a_2b_2 + a_3b_3$

Hence $a_1b_1 + a_2b_2 + a_3b_3 = |\underline{a}||\underline{b}|cos\theta$

(3) $\underline{a}.\underline{b} = 0 \Rightarrow \underline{a}$ and $\underline{b}$ are perpendicular (assuming that they aren't zero)

(4) One application of the scalar product is in determining the work done (W) by a force $\underline{F}$, which has component $|\underline{F}|cos\theta$ in the direction of the displacement $\underline{s}$ of the object in question, so that $W = \underline{F}.\underline{s}$

(5) Example

A triangle is made up of the points O$(0,0)$, A$(-3,4)$ and B $(12,5)$.

Find the angle between the sides OA and OB

$$cos\theta = \frac{\overrightarrow{OA}.\overrightarrow{OB}}{|\overrightarrow{OA}.||\overrightarrow{OB}|}$$

$\overrightarrow{OA}.\overrightarrow{OB} = \binom{-3}{4}.\binom{12}{5} = (-3)(12) + (4)(5) = -16$

$\left|\overrightarrow{OA}\right| = \sqrt{(-3)^2 + 4^2} = 5$ & $\left|\overrightarrow{OB}\right| = \sqrt{12^2 + 5^2} = 13$

Hence $cos\theta = \frac{-16}{(5)(13)} = -0.24615$ and $\theta = 104.2°$

[Note: The scalar product often provides a more convenient alternative to using the Cosine rule.]

Note: In this example, the angle between the sides OA and OB is obtuse, and so 104.2° is the appropriate answer. In other situations, where the angle between two lines is to be found, the acute angle is usually given.

(6) **Exercise 1**

If $\overrightarrow{AB} = \binom{1}{-2}$ & $\overrightarrow{CD} = \binom{-1}{1}$, find the angle θ between $\overrightarrow{AB}$ & $\overrightarrow{CD}$, and the angle ϕ between $\overrightarrow{AB}$ & $\overrightarrow{DC}$.

Solution

$\overrightarrow{AB}.\overrightarrow{CD} = \left|\overrightarrow{AB}\right|\left|\overrightarrow{CD}\right|cos\theta$

$\Rightarrow \binom{1}{-2}.\binom{-1}{1} = \sqrt{5}\sqrt{2}cos\theta$

$\Rightarrow cos\theta = \frac{-1+(-2)}{\sqrt{10}} = \frac{-3}{\sqrt{10}}$

$\Rightarrow \theta = 161.6°$ (to 1dp)

$\overrightarrow{AB}.\overrightarrow{DC} = \left|\overrightarrow{AB}\right|\left|\overrightarrow{DC}\right|cos\phi$

$\Rightarrow \binom{1}{-2}.\binom{1}{-1} = \sqrt{5}\sqrt{2}cos\phi$

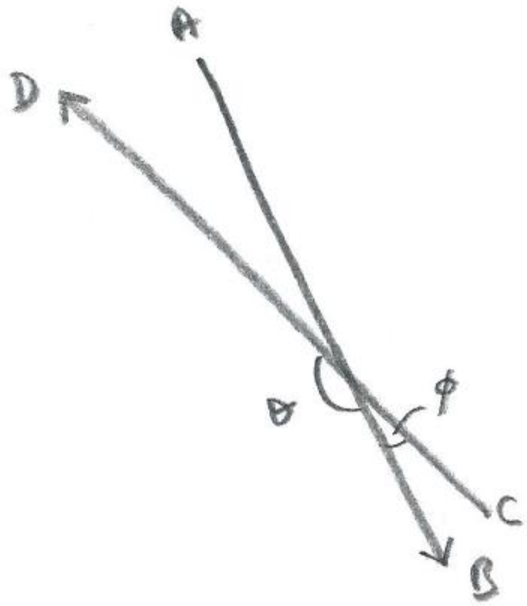
$\Rightarrow cos\phi = \frac{1+2}{\sqrt{10}} = \frac{3}{\sqrt{10}}$

$\Rightarrow \phi = 18.4°$ (to 1dp)

Note that $\cos(180 - \theta) = -cos\theta$, so that $cos\phi = -cos\theta$

$\Rightarrow \phi = 180 - \theta$ (in this context)

Thus, reversing the direction of one of the lines has the effect of changing the angle from θ to its supplement $180 - \theta$ (see diagram below).



Exercise 2

Find the angle between the lines $2x - y = 3$ and $x + 4y = 5$

Solution

The gradients of the two lines are 2 and $-\frac{1}{4}$

Their direction vectors are $\binom{1}{2}$ & $\binom{4}{-1}$

$$\binom{1}{2} \cdot \binom{4}{-1} = \left|\binom{1}{2}\right| \left|\binom{4}{-1}\right| cos\theta$$

so that $cos\theta = \frac{4-2}{\sqrt{5}\sqrt{17}} = 0.21693$

and $\theta = 77.471 = 77.5° \ (1dp)$

(7) Useful devices

(i) $\underline{a}.\underline{a} = |\underline{a}|^2$

(ii) If $\underline{r}.\underline{a} = 0$ for all $\underline{r}$, then $\underline{a} = 0$

(8) Consider a parallelogram with sides $\underline{u}$ & $\underline{v}$. Then , if the diagonals of the parallelogram are perpendicular,

$(\underline{u} + \underline{v}).(\underline{u} - \underline{v}) = 0$

Expanding this gives $\underline{u}.\underline{u} = \underline{v}.\underline{v}$; ie $|\underline{u}|^2 = |\underline{v}|^2$, so that the two sides are equal; ie the parallelogram is a rhombus.