STEP/Vectors Q8 (30/6/23)

Given that $\boldsymbol{a}$, $\boldsymbol{b}$ & $\boldsymbol{c}$ are linearly independent vectors, establish whether the vectors $\boldsymbol{a}+\boldsymbol{b}$, $\boldsymbol{a}-\boldsymbol{c}$ & $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$ are linearly independent.

Solution

Method 1

Suppose that $\alpha(\boldsymbol{a}+\boldsymbol{b})+\beta(\boldsymbol{a}-\boldsymbol{c})+\gamma(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})=0$

Then $(\alpha+\beta+\gamma)\boldsymbol{a}+(\alpha+\gamma)\boldsymbol{b}+(-\beta+\gamma)\boldsymbol{c}=0$

As $\boldsymbol{a}$, $\boldsymbol{b}$ & $\boldsymbol{c}$ are linearly independent,

$\alpha+\beta+\gamma=0$

$\alpha+\gamma=0$

$-\beta+\gamma=0$

giving $\alpha+\beta=0$

and hence $\gamma=0$ and so $\alpha=\beta=0$ also,

and so $\boldsymbol{a}+\boldsymbol{b}$, $\boldsymbol{a}-\boldsymbol{c}$ & $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$ are linearly independent.

Method 2

$\boldsymbol{a}$, $\boldsymbol{b}$ & $\boldsymbol{c}$ are linearly independent vectors $\Rightarrow |\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}| = 0$

$\Rightarrow |\boldsymbol{a}+\boldsymbol{b},\ \boldsymbol{b},\ \boldsymbol{c}+(\boldsymbol{a}+\boldsymbol{b})| = 0$

[since $|\boldsymbol{a}+\boldsymbol{b},\ \boldsymbol{b},\ \boldsymbol{c}| = |\boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{c}|$:

As an example, consider

$\begin{vmatrix} p+r & r \\ q+s & s \end{vmatrix} = (p+r)s-(q+s)r = (ps-qr)+rs-sr$

$= \begin{vmatrix} p & r \\ q & s \end{vmatrix}$; also $|k\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}\ | = k|\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}\ |$]

Then, if $\boldsymbol{a}+\boldsymbol{b}$, $\boldsymbol{a}-\boldsymbol{c}$ & $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$ are in fact linearly independent, we want to be able to obtain $\boldsymbol{a}-\boldsymbol{c}$ by adding multiples of $\boldsymbol{a}+\boldsymbol{b}$ & $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$ to **b.** In fact, because **b** can be replaced with k**b**, we can look for a relation of the form:

$$\boldsymbol{a}-\boldsymbol{c}=k\boldsymbol{b}+\lambda(\boldsymbol{a}+\boldsymbol{b})+\mu(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})$$

Equating coefficients of the linearly independent **a, b & c:**

$1=\lambda+\mu;\ \ 0=k+\lambda+\mu; -1=\mu$

So $\mu=-1,\ \lambda=2$ & $k=-1$

Thus $|\boldsymbol{a}+\boldsymbol{b},\ \boldsymbol{a}-\boldsymbol{c},\ \boldsymbol{c}+(\boldsymbol{a}+\boldsymbol{b})|=0$,

so that the 3 vectors are linearly independent