STEP/Vectors Q7 (30/6/23)

Find the angle between adjacent sloping faces of a right square-based pyramid, where the faces are equilateral triangles (as shown in Figure 1).

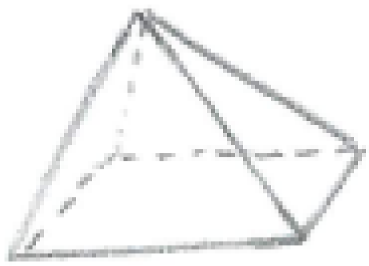

Figure 1

Solution

Without loss of generality, we can assume that the sides of the equilateral triangles forming the faces have length 2. The medians of the equilateral triangles leading to the vertex then have length $\sqrt{3}$.

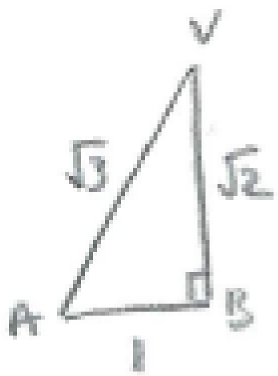

Figure 2

Referring to figure 2, we can form the right-angled triangle with corners at the vertex (V), the centre of the base of the pyramid (B), and the base of a median (A). By Pythagoras, $VB = \sqrt{2}$.

Create x and y axes along the bottom of two adjacent sloping faces of the pyramid, with z being vertical (so that the origin is at one corner of the base of the pyramid).

Then a vector equation of the plane containing one face and the y-axis is:

$$\underline{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix},$$

since the plane contains the origin, the point (0,2,0) and the point $(1,1,\sqrt{2})$, which is V (these are the 3 corners of the face).

Converting to a Cartesian equation, we have

$x = \mu\,, y = 2\lambda + \mu$ and $z = \mu\sqrt{2}$

so that $z = x\sqrt{2}$, and the equation can be written as $\sqrt{2}x + 0y - z = 0$.

Hence this face of the pyramid has direction vector $\begin{pmatrix} \sqrt{2} \\ 0 \\ -1 \end{pmatrix}$

or $\begin{pmatrix} -\sqrt{2} \\ 0 \\ 1 \end{pmatrix}$, to ensure that it is pointing away from the inside of the pyramid.

Similarly, the direction vector of the plane containing one face and the x-axis is $\begin{pmatrix} 0 \\ -\sqrt{2} \\ 1 \end{pmatrix}$.

The angle between the outward-pointing direction vectors of these faces is then given by

$$cos\theta = \frac{\begin{pmatrix} -\sqrt{2} \\ 0 \\ 1 \end{pmatrix} . \begin{pmatrix} 0 \\ -\sqrt{2} \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} -\sqrt{2} \\ 0 \\ 1 \end{pmatrix}\right| \left|\begin{pmatrix} 0 \\ -\sqrt{2} \\ 1 \end{pmatrix}\right|} = \frac{1}{3}$$

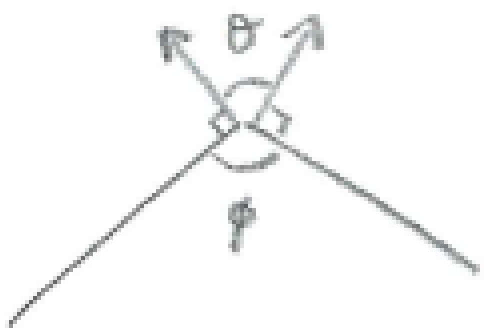

Figure 3

Referring to Figure 3, the angle that we require is $\phi = 180 - \theta$, and $cos\phi = -cos\theta = -\frac{1}{3}$

Thus $\phi = cos^{-1}\left(-\frac{1}{3}\right) = 109.5°$ (1dp)