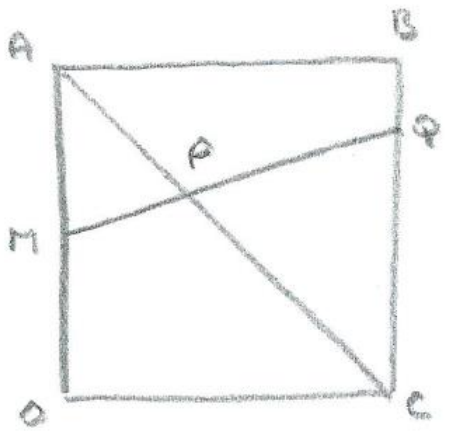
STEP/Vectors Q6 (30/6/23)

In the diagram below, OABC is a square, M is the midpoint of OA, BQ is a quarter of BC, and P is the intersection of AC and MQ.



If $\underline{a} = \overrightarrow{OA}$ and $\underline{c} = \overrightarrow{OC}$, show that $\overrightarrow{OP} = \frac{3}{5}\underline{a} + \frac{2}{5}\underline{c}$

Solution

Let $\overrightarrow{OP} = \alpha\underline{a} + \gamma\underline{c}$

To take account of the fact that P lies on AC, we can write:

$\overrightarrow{AP} = \lambda\overrightarrow{AC}$,

so that $\overrightarrow{OP} - \overrightarrow{OA} = \lambda(\overrightarrow{OC} - \overrightarrow{OA})$

and $\alpha\underline{a} + \gamma\underline{c} - \underline{a} = \lambda\underline{c} - \lambda\underline{a}$

or $(\alpha - 1 + \lambda)\underline{a} = (\lambda - \gamma)\underline{c}$

Then, as $\underline{a}$ and $\underline{c}$ aren't parallel, the only possibility is that

$\alpha - 1 + \lambda = 0$ and $\lambda - \gamma = 0$

so that $\alpha - 1 + \gamma = 0$ (1)

Similarly, to take account of the fact that P lies on MQ, we can write: $\overrightarrow{MP} = \mu\overrightarrow{MQ}$,

so that $\overrightarrow{OP} - \overrightarrow{OM} = \mu(\overrightarrow{MO} + \overrightarrow{OC} + \overrightarrow{CQ})$

and $\alpha\underline{a} + \gamma\underline{c} - \frac{1}{2}\underline{a} = \mu(-\frac{1}{2}\underline{a} + \underline{c} + \frac{3}{4}\underline{a})$

or $\left(\alpha - \frac{1}{2} - \frac{1}{4}\mu\right)\underline{a} = (\mu - \gamma)\underline{c}$

and so, as before, $\alpha - \frac{1}{2} - \frac{1}{4}\mu = 0$ and $\mu - \gamma = 0$,

giving $\alpha - \frac{1}{2} - \frac{1}{4}\gamma = 0$ (2)

Then, subtracting (2) from (1) gives

$-\frac{1}{2} + \frac{5}{4}\gamma = 0$, so that $\gamma = \frac{2}{5}$, and $\alpha = 1 - \gamma = \frac{3}{5}$

and $\overrightarrow{OP} = \frac{3}{5}\underline{a} + \frac{2}{5}\underline{c}$, as required.