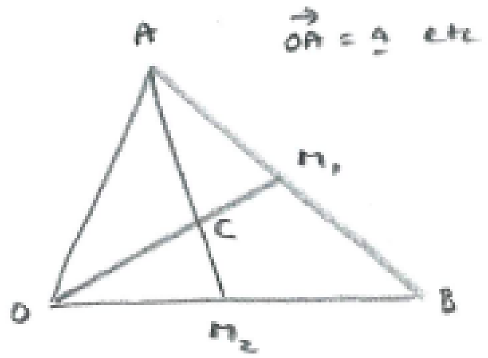
STEP/Vectors Q4 (30/6/23)

Prove that the centre of mass of a triangular lamina lies 2/3 of the way along any of the medians.

Solution



Let $\overrightarrow{OC} = \lambda\overrightarrow{OM_1}$ & $\overrightarrow{AC} = \mu\overrightarrow{AM_2}$ (1)

[standard technique: represents the fact that C lies on the line OM_1]

Also, $\overrightarrow{OC} = \underline{a} + \overrightarrow{AC}$ (2) [standard technique: 2 ways of getting to the same place]

Substitute (1) into (2) ⇒ $\lambda\overrightarrow{OM_1} = \underline{a} + \mu\overrightarrow{AM_2}$ (3)

Now, $\overrightarrow{OM_1} = ½ (\underline{a} + \underline{b})$ & $\overrightarrow{AM_2} = ½ \underline{b} - \underline{a}$ (4)

Substitute (4) into (3) ⇒ $½ \lambda (\underline{a} + \underline{b}) = \underline{a} + \mu (½ \underline{b} - \underline{a})$ (5)

$\Rightarrow (\frac{\lambda}{2} + \mu - 1)\,\underline{a} + (\frac{\lambda}{2} - \frac{\mu}{2})\,\underline{b} = 0$

Provided $\underline{a}$ & $\underline{b}$ are not parallel, there is only one way of expressing a vector as a combination of $\underline{a}$ & $\underline{b}$

In this case, $\frac{\lambda}{2} + \mu - 1 = 0$ & $\frac{\lambda}{2} - \frac{\mu}{2} = 0$ (6)

[standard technique: equivalent to equating coefficients of $\underline{a}$ & of $\underline{b}$ in (5)]

Then (6) $\Rightarrow \lambda = \mu = \frac{2}{3}$

ie the centre of mass lies two-thirds of the way along any of the medians, from the relevant vertex.