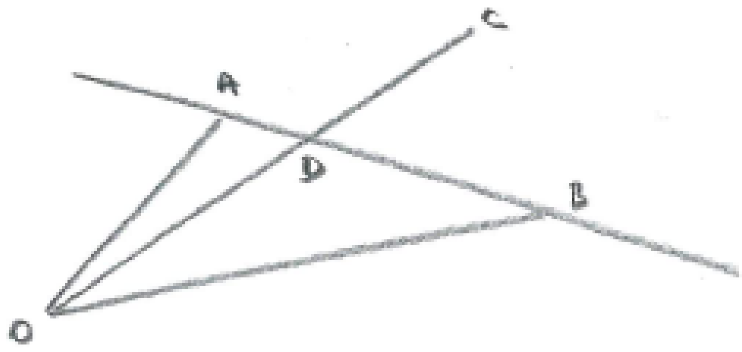
STEP - Vectors



(1) $\underline{d} = \underline{a} + k\overrightarrow{AB}$

(2) $\underline{d} = (1-\lambda)\underline{a} + \lambda\underline{b}$

(3) $\underline{d} = \mu\underline{c}$

(4) $p\underline{a} + q\underline{b} = r\underline{a} + s\underline{b} \Rightarrow p = r \;\&\; q = s$

Show that if $|\underline{a} - \underline{b}| = |\underline{a} + \underline{b}|$, then $\underline{a}$ & $\underline{b}$ are perpendicular (for non-zero $\underline{a}$ & $\underline{b}$).

Solution

$|\underline{a}-\underline{b}| = |\underline{a}+\underline{b}| \Rightarrow |\underline{a}-\underline{b}|^2 = |\underline{a}+\underline{b}|^2$

$\Rightarrow (\underline{a}-\underline{b}).(\underline{a}-\underline{b}) = (\underline{a}+\underline{b}).(\underline{a}+\underline{b})$

[$\underline{x}.\underline{x} = |\underline{x}|\,|\underline{x}|cos0° = |\underline{x}|^2$]

$\Rightarrow \underline{a}.\underline{a} - \underline{a}.\underline{b} - \underline{b}.\underline{a} + \underline{b}.\underline{b} = \underline{a}.\underline{a} + \underline{a}.\underline{b} + \underline{b}.\underline{a} + \underline{b}.\underline{b}$

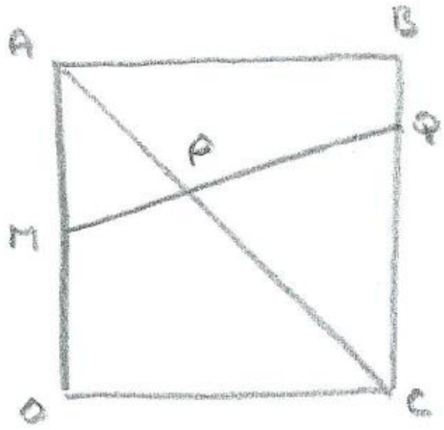
$\Rightarrow -2\underline{a}.\underline{b} = 2\underline{a}.\underline{b}$ [since $\underline{a}.\underline{b} = \underline{b}.\underline{a}$]

$\Rightarrow \underline{a}.\underline{b} = 0$

and hence $\underline{a}$ & $\underline{b}$ are perpendicular

[Geometrically, $|\underline{a}-\underline{b}|$&$|\underline{a}+\underline{b}|$ are the 'short' and 'long' diagonals of the parallelogram formed from the adjacent sides $\underline{a}$ & $\underline{b}$. When these diagonals are equal, the parallelogram is a rectangle.]

In the diagram below, OABC is a square, M is the midpoint of OA, BQ is a quarter of BC, and P is the intersection of AC and MQ.



If $\underline{a} = \overrightarrow{OA}$ and $\underline{c} = \overrightarrow{OC}$, show that $\overrightarrow{OP} = \frac{3}{5}\underline{a} + \frac{2}{5}\underline{c}$

Solution

Let $\overrightarrow{OP} = \alpha\underline{a} + \gamma\underline{c}$

To take account of the fact that P lies on AC, we can write:

$\overrightarrow{AP} = \lambda\overrightarrow{AC}$,

so that $\overrightarrow{OP} - \overrightarrow{OA} = \lambda(\overrightarrow{OC} - \overrightarrow{OA})$

and $\alpha\underline{a} + \gamma\underline{c} - \underline{a} = \lambda\underline{c} - \lambda\underline{a}$

or $(\alpha - 1 + \lambda)\underline{a} = (\lambda - \gamma)\underline{c}$

Then, as $\underline{a}$ and $\underline{c}$ aren't parallel, the only possibility is that

$\alpha - 1 + \lambda = 0$ and $\lambda - \gamma = 0$

so that $\alpha - 1 + \gamma = 0$ (1)

Similarly, to take account of the fact that P lies on MQ, we can write: $\overrightarrow{MP} = \mu\overrightarrow{MQ}$,

so that $\overrightarrow{OP} - \overrightarrow{OM} = \mu(\overrightarrow{MO} + \overrightarrow{OC} + \overrightarrow{CQ})$

and $\alpha\underline{a} + \gamma\underline{c} - \frac{1}{2}\underline{a} = \mu(-\frac{1}{2}\underline{a} + \underline{c} + \frac{3}{4}\underline{a})$

or $\left(\alpha - \frac{1}{2} - \frac{1}{4}\mu\right)\underline{a} = (\mu - \gamma)\underline{c}$

and so, as before, $\alpha - \frac{1}{2} - \frac{1}{4}\mu = 0$ and $\mu - \gamma = 0$,

giving $\alpha - \frac{1}{2} - \frac{1}{4}\gamma = 0$ (2)

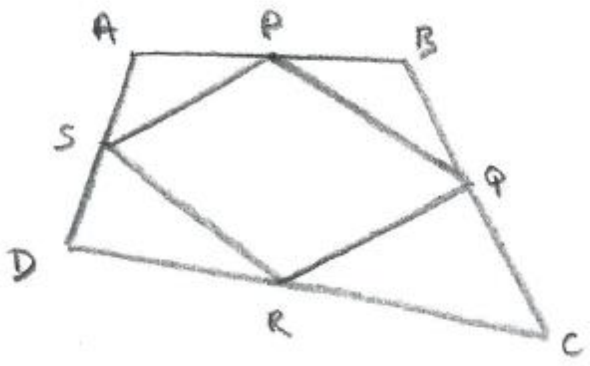
Then, subtracting (2) from (1) gives

$-\frac{1}{2} + \frac{5}{4}\gamma = 0$, so that $\gamma = \frac{2}{5}$, and $\alpha = 1 - \gamma = \frac{3}{5}$

and $\overrightarrow{OP} = \frac{3}{5}\underline{a} + \frac{2}{5}\underline{c}$, as required.

Use vectors to prove that the mid-points of the sides of any quadrilateral form the vertices of a parallelogram.

Solution



Referring to the diagram (where $\underline{a} = \overrightarrow{OA}\ \ etc$),

$\underline{q} - \underline{p} = \frac{1}{2}(\underline{b} + \underline{c}) - \frac{1}{2}(\underline{a} + \underline{b}) = \frac{1}{2}(\underline{c} - \underline{a})$

and $\underline{r} - \underline{s} = \frac{1}{2}(\underline{c} + \underline{d}) - \frac{1}{2}(\underline{a} + \underline{d}) = \frac{1}{2}(\underline{c} - \underline{a}) = \underline{q} - \underline{p}$

So the sides PQ & SR are of equal length and parallel.

This means that $PQRS$ is a parallelogram.