STEP/Trigonometry Q5 (30/6/23)

Show that $\arctan\left(\frac{1+a}{\sqrt{1-a^2}}\right) - \arctan\left(\frac{a}{\sqrt{1-a^2}}\right) = \arctan\left(\frac{\sqrt{1-a}}{\sqrt{1+a}}\right)$

Solution

Consider $\theta - \phi$, where $tan\theta = A$ & $tan\phi = B$

Then $tan(\theta - \phi) = \frac{A-B}{1+AB}$

With $A = \frac{1+a}{\sqrt{1-a^2}}$ and $B = \frac{a}{\sqrt{1-a^2}}$,

$\arctan\left(\frac{1+a}{\sqrt{1-a^2}}\right) - \arctan\left(\frac{a}{\sqrt{1-a^2}}\right) = \theta - \phi = arctan(\frac{A-B}{1+AB})$

and $\frac{A-B}{1+AB} = \frac{[(1+a)-a]\sqrt{1-a^2}}{(1-a^2)+(1+a)a} = \frac{\sqrt{1-a^2}}{1+a} = \frac{\sqrt{1-a}\sqrt{1+a}}{1+a} = \frac{\sqrt{1-a}}{\sqrt{1+a}}$,

giving the required result.