STEP/Trigonometry Q11 (30/6/23)

Prove that $\frac{1+cosec\theta}{cot\theta} = \frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}$

Solution

Let $\phi = \frac{\theta}{2}$, so that $\frac{1+cosec\theta}{cot\theta} = \frac{1+\tan(\frac{\theta}{2})}{1-\tan(\frac{\theta}{2})}$ is equivalent to

$\frac{1+cosec(2\phi)}{\cot(2\phi)} = \frac{1+\tan\phi}{1-\tan\phi}$ (1)

Now LHS of (1) $= \frac{\sin(2\phi)+1}{\cos(2\phi)} = \frac{2sin\phi cos\phi+1}{cos^2\phi-sin^2\phi}$ (3)

And RHS of (2) $= \frac{cos\phi+sin\phi}{cos\phi-sin\phi} = \frac{(cos\phi+sin\phi)^2}{cos^2\phi-sin^2\phi} = \frac{cos^2\phi+sin^2\phi+2sin\phi cos\phi}{cos^2\phi-sin^2\phi}$

$= \frac{1+2sin\phi cos\phi}{cos^2\phi-sin^2\phi}$, which equals (3)