

STEP/Transformations Q3 (28/6/23)

(i) Find a series of transformations that can be applied to $y = \frac{1}{x}$ to produce $y = \frac{3x-2}{6x-1}$.

(ii) Sketch the curve $y = \frac{3x-2}{6x-1}$.

Solution

$$(i) \frac{3x-2}{6x-1} = \frac{3x-\frac{1}{2}-\frac{3}{2}}{6x-1} = \frac{1}{2} - \frac{3}{12} \left(\frac{1}{x-\frac{1}{6}} \right)$$

So a possible series of transformations is:

a translation of $\begin{pmatrix} 1 \\ 6 \\ 0 \end{pmatrix}$,

followed by a stretch of scale factor $\frac{1}{4}$ in the y -direction,

followed by a reflection in the x -axis,

followed by a translation of $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

[Note: $\frac{1}{2} - \frac{3}{12} \left(\frac{1}{x-\frac{1}{6}} \right) = \frac{1}{2} - \frac{1}{4x-\frac{2}{3}}$, so an alternative series of transformations is:

a translation of $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ [$\frac{1}{x} \rightarrow \frac{1}{x-\frac{2}{3}}$] followed by

a stretch of scale factor $\frac{1}{4}$ in the x -direction [$\frac{1}{x-\frac{2}{3}} \rightarrow \frac{1}{4x-\frac{2}{3}}$], followed

by a reflection in the x -axis, followed by a translation of $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

Alternatively, $\frac{1}{4x-\frac{2}{3}}$ could be obtained instead by a stretch of scale

factor $\frac{1}{4}$ in the x -direction [$\frac{1}{x} \rightarrow \frac{1}{4x}$] (or a stretch of scale factor $\frac{1}{4}$ in the y -direction

[$\frac{1}{x} \rightarrow \frac{1}{4} \left(\frac{1}{x} \right)$]), followed by a translation of $\begin{pmatrix} 1 \\ 6 \\ 0 \end{pmatrix}$ [$\frac{1}{4x} \rightarrow \frac{1}{4(x-\frac{1}{6})}$].]

(ii) As an alternative to performing the transformations in (i):

$$\text{Step 1: } x = 0 \Rightarrow y = 2 ; y = 0 \Rightarrow x = \frac{2}{3}$$

$$\text{Step 2: vertical asymptote when } 6x - 1 = 0 \Rightarrow x = \frac{1}{6}$$

$$x = \frac{1}{6} + \delta \text{ (} \delta > 0 \text{ is small)} \Rightarrow y = \frac{3x-2}{6x-1} = \frac{-}{+}; \text{ ie } y < 0$$

$$[x = \frac{1}{6} - \delta \Rightarrow y = \frac{-}{-}; \text{ ie } y > 0]$$

$$\text{Step 3: } \lim_{x \rightarrow \infty} \frac{3x-2}{6x-1} = \lim_{x \rightarrow \infty} \frac{3-\frac{2}{x}}{6-\frac{1}{x}} = \frac{3}{6} = \frac{1}{2} \text{ (and also as } x \rightarrow -\infty)$$

$$\text{Step 4: When } x = 100, y = \frac{298}{599} < \frac{1}{2}, \text{ so that } y \rightarrow \frac{1}{2}^- \text{ as } x \rightarrow \infty$$

$$\text{and when } x = -100, y = \frac{-302}{-601} > \frac{1}{2}, \text{ so that } y \rightarrow \frac{1}{2}^+ \text{ as } x \rightarrow -\infty$$

