

STEP/Statistics Q1 (14/6/23)

Show that 1 standard deviation to either side of the mean of the Normal distribution occurs at the point of inflexion of the Normal curve.

Solution

Considering $N(0, 1)$, $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

$$\phi'(x) = \frac{-x}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$\text{and } \phi''(x) = \frac{-1}{\sqrt{2\pi}} \{e^{-\frac{1}{2}x^2} + x(-x)e^{-\frac{1}{2}x^2}\}$$

A point of inflexion is a turning point of the gradient, for which a necessary condition is that the gradient is stationary; ie

$$\frac{d}{dx} \phi'(x) = 0 \text{ or } \phi''(x) = 0$$

[Technically, to confirm that it is a turning point of the gradient, we should check that $\frac{d^2}{dx^2} \phi'(x) \neq 0$; ie $\phi'''(x) \neq 0$ (this is a sufficient condition; a necessary condition is that the first non-zero derivative of $\phi'(x)$ is an even derivative). However, we can see from the curve that there is a point of inflexion.]

$$\phi''(x) = 0 \Rightarrow 1 - x^2 = 0 \Rightarrow x = \pm 1, \text{ as required.}$$