STEP/Sequences & Series Q5 (27/6/23)

(i) Consider the sequence defined by $u_n = au_{n-1} + b$,

where a & b are real constants, and u_0 is given.

What familiar sequences are special cases of this sequence?

Solution

Setting $a = 1$ gives an arithmetic sequence.

Setting $b = 0$ gives a geometric sequence.

(ii) Define a new sequence by $v_n = u_n + c$

For what value of c, in terms of a & b, will v_n be a geometric sequence? For what value of a does this not work?

Solution

$v_{n-1} = u_{n-1} + c$, and hence

$u_n = au_{n-1} + b \Rightarrow v_n - c = a(v_{n-1} - c) + b$

$\Rightarrow v_n = av_{n-1} + b + c(1 - a)$

For v_n to be a geometric sequence, we want $b + c(1 - a) = 0$,

so that $c = \frac{-b}{1-a} = \frac{b}{a-1}$, provided that $a \neq 1$

When $a = 1$, u_n, and hence v_n also, are arithmetic sequences.

(iii) If $u_n = 2u_{n-1} + 3$, and $u_0 = 4$, find a formula for u_n in terms of n

Solution

From (ii), $c = \frac{3}{2-1} = 3$ and $v_n = 2v_{n-1}$

Then $v_n = v_0(2^n)$

and $v_n = u_n + 3$, so that $u_n + 3 = (u_0 + 3)(2^n)$

and $\therefore u_n = 7(2^n) - 3$

(and this can be checked by comparing with $u_n = 2u_{n-1} + 3$, and $u_0 = 4$)

(iv) Find a similar formula for $u_n = au_{n-1} + b$, where u_0 is given.

Solution

From (ii), $c = \frac{b}{a-1}$ and $v_n = av_{n-1}$

Then $v_n = v_0(a^n)$

and $v_n = u_n + c$, so that $u_n + c = (u_0 + c)(a^n)$

and $\therefore u_n = (u_0 + c)(a^n) - c = \left(u_0 + \frac{b}{a-1}\right)(a^n) - \frac{b}{a-1}$

(v) Under what conditions will u_n be constant? Give a non-trivial example.

Solution

Either $a = 1$ & $b = 0$

Or $a = 0$ and $u_0 = b$

Or $u_0 + \frac{b}{a-1} = 0$; ie $u_0 = \frac{b}{1-a}$

For example, $u_n = 2u_{n-1} - 1$, where $u_0 = 1$