STEP/Sequences & Series Q3 (27/6/23)

Show that $\sum_{r=0}^{n} \binom{n}{r} = 2^n$

Solution

Method 1: Consider the Binomial expansion of $(1+1)^n$

Method 2: Pascal's triangle:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

Result to prove: the sum of each row is twice the sum of the previous one, so that the sum of each row is a power of 2. (*)

Initial result to prove: $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$

Proof

If r items are to be chosen from n items, then either the 1st item is included or it isn't.

If it is included, then there are $\binom{n-1}{r-1}$ ways of choosing the remaining $r-1$ items that are required.

If it isn't included, then there are $\binom{n-1}{r}$ ways of choosing the remaining r items that are required.

This gives a total of $\binom{n-1}{r-1} + \binom{n-1}{r}$ ways of choosing the r items.

This result shows that each entry, apart from the 1s at each end, is the sum of the two entries above (to the left and right), and hence each entry in row $n-1$, apart from the 1s on each side, contributes to two entries in row n. The 1s in row $n-1$ each contribute to only one entry in row n, but the 1s at each end of row n make up the shortfall, so that each entry in row $n-1$ is represented twice in the sum for row n. This proves result (*), and hence that $\sum_{r=0}^{n}\binom{n}{r}=2^n$.

Method 3: Counting ways of selecting any number of items (ie 0, 1, 2, ... or n)

1st counting method: $\sum_{r=0}^{n}\binom{n}{r}$

2nd counting method: For each object, there are 2 choices: include or exclude; giving 2^n possibilities

Method 4: Induction

If true for $n=k$, so that $\sum_{r=0}^{k}\binom{k}{r}=2^k$,

then $\sum_{r=0}^{k+1}\binom{k+1}{r}=\binom{k+1}{0}+\{\sum_{r=1}^{k}\binom{k+1}{r}\}+\binom{k+1}{k+1}$

$=1+\sum_{r=1}^{k}\left\{\binom{k}{r-1}+\binom{k}{r}\right\}+1$

$=1+\{\sum_{r-1=0}^{k-1}\binom{k}{r-1}\}+[\{\sum_{r=0}^{k}\binom{k}{r}\}-\binom{k}{0}]+1$

$=1+\{\sum_{R=0}^{k-1}\binom{k}{R}\}+[2^k-1]+1$

$= 1 + \{\sum_{R=0}^{k} \binom{k}{R}\} - \binom{k}{k} + 2^k$

$= 1 + 2^k - 1 + 2^k = 2^{k+1}$