STEP/Sequences & Series Q2 (27/6/23)

‘Perfect powers’ can be defined as follows:

m^k for integer $m \geq 2$ & integer $k \geq 2$

Prove that the sum of the reciprocals of all perfect powers is 1 (including duplicates; eg $4^2 = 2^4$).

Solution

$$\sum_{m=2}^{\infty}\sum_{k=2}^{\infty}\frac{1}{m^k}=\sum_{m=2}^{\infty}\frac{\frac{1}{m^2}}{1-\frac{1}{m}}=\sum_{m=2}^{\infty}\frac{1}{m(m-1)}=\sum_{m=2}^{\infty}\left(\frac{1}{m-1}-\frac{1}{m}\right)$$

$$=\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots\right)-\left(\frac{1}{2}+\frac{1}{3}+\cdots\right)=1$$