If and Only If Proofs (STEP) (6 pages; 18/5/23)

(1) General Example: (Given two statements A & B) Prove that $A \Leftrightarrow B$

Possible approaches:

(i) Prove that $A \Leftrightarrow C \Leftrightarrow D \ldots \Leftrightarrow B$

(ii) Prove that $A \Rightarrow B$ and $B \Rightarrow A$

(iii) Prove that $A \Rightarrow B$ and $A' \Rightarrow B'$ (this is equivalent to $B \Rightarrow A$)

(iv) Break down into cases; eg $C_1: x < 0, C_2: x = 0$ & $C_3: x > 0$

(especially where B is a composite statement, such as "C and either D or E")

Then, for each case C_i, prove either that A is true and B is true, or that A is false and B is false.

Notes

(a) For each case, we can investigate whether A (or B) turns out to be true or false, and then show that B (or A) follows suit.

(b) In some situations, it is possible to enumerate all possible situations, but it is obviously desirable to keep the number of cases to a minimum.

(c) If B (say) is a composite statement, then it may be advantageous to base the classification into cases on this statement (eg B may involve "$x > 0$", and this may suggest the cases $C_1: x < 0, C_2: x = 0$ & $C_3: x > 0$)

(d) It may not be possible to arrange the cases so that A is always true, or always false. Instead we may have to be satisfied with having narrowed things down so that, for a particular case (eg with the knowledge that $x > 0$), we still need to show that $A \Leftrightarrow B$ (whilst for other cases, we need only show that $C_i \Rightarrow A$ and

$C_i \Rightarrow B$).

(e) When we show that $C_i \Rightarrow A$ and $C_i \Rightarrow B$, we are effectively doing the following:

$A \,\&\, C_i \Rightarrow C_i \Rightarrow B \Rightarrow B \,\&\, C_i$ or $A' \,\&\, C_i \Rightarrow C_i \Rightarrow B' \Rightarrow B' \,\&\, C_i$

and $B \,\&\, C_i \Rightarrow C_i \Rightarrow A \Rightarrow A \,\&\, C_i$ or $B' \,\&\, C_i \Rightarrow C_i \Rightarrow A' \Rightarrow A' \,\&\, C_i$

ie we are showing, case by case, that $A \Rightarrow B$ and $B \Rightarrow A$.

(2) Example: Given that $\underline{a}$, $\underline{b}$ & $\underline{c}$ are non-zero position vectors, and that the angles between $\underline{a}$ & $\underline{c}$ and $\underline{b}$ & $\underline{c}$ are α & β respectively, prove that $\underline{c}$ bisects the angle between $\underline{a}$ & $\underline{b}$ if and only if $\alpha = \beta$ and either $\underline{a} \neq k\, \underline{b}$ or $\alpha = 0$

Solution

Note that $0 \leq \alpha, \beta \leq \frac{\pi}{2}$.

Let S be the event "$\underline{c}$ bisects the angle between $\underline{a}$ & $\underline{b}$".

Let T_1 be the event "$\alpha = \beta$".

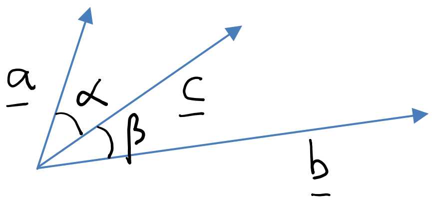
Let T_2 be the event "$\underline{a} \neq k\, \underline{b}$"

Let T_3 be the event "$\alpha = 0$"

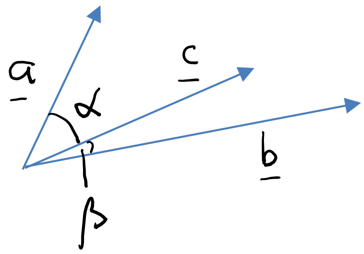
Let T be the event T_1 and either T_2 or T_3.

One of the following 4 possible types of situation must occur:

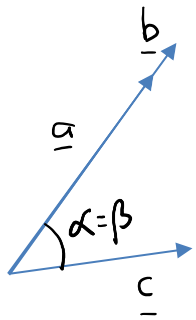
A



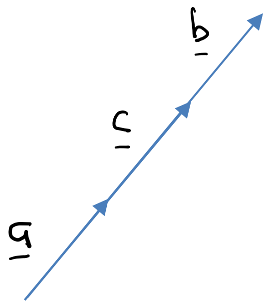
B



C



D



[These have been chosen so that, in each case, either S will always occur, or S will never occur.]

We then need to show that, in each case, where S occurs, T occurs as well; and where S doesn't occur, T doesn't occur either. (*)

For A, S occurs. Also, T_1 and T_2 occur, and so T occurs.

For B, S doesn't occur. Also, T_1 doesn't occur, and so T doesn't occur.

For C, S doesn't occur. Also, T_1 occurs, but neither T_2 nor T_3 occurs, and so T doesn't occur.

For D, S occurs. Also, T_1 occurs, and although T_2 doesn't occur, T_3 does occur, and so T occurs.

So we have shown (*).

[Note that, with this approach, once we have chosen the classification of the events (ie into 4 cases here), we just need to show in each case C either:

that $C \Rightarrow S$ and $C \Rightarrow T$, or that $C \Rightarrow S'$ and $C \Rightarrow T'$]

(3) Example: Prove that $\frac{1}{a} > \frac{1}{b}$ if and only if $a < b < 0$ or $0 < a < b$ or $a > 0$ & $b < 0$

[This is easily seen from the graph of $y = \frac{1}{x}$.]

Solution

Note: Clearly the problem is undefined for $a = 0$ or $b = 0$.

Denote $\frac{1}{a} > \frac{1}{b}$ by S; $a < b < 0$ by T_1; $0 < a < b$ by T_2; $a > 0$ & $b < 0$ by T_3, and "T_1 or T_2 or T_3" by T.

[In theory, we could consider the 8 cases arising from the classification: $a <$ or $> 0, b <$ or $> 0, a <$ or $> b$, but this could be time-consuming.]

Consider the following cases:

$C_1: a > 0, b > 0$

$C_2: a < 0, b < 0$

$C_3: a > 0, b < 0$

$C_4: a < 0, b > 0$

Case C_1

$S: \frac{1}{a} > \frac{1}{b} \Rightarrow b > a$ (as $a > 0, b > 0$) $\Rightarrow T_2 \Rightarrow T$

and $T \Rightarrow T_2$ (as $a > 0, b > 0$) $\Rightarrow b > a \Rightarrow \frac{1}{a} > \frac{1}{b}$ (as $a > 0, b > 0$)

So, for case C_1, $S \Leftrightarrow T$.

Case C_2

$S: \frac{1}{a} > \frac{1}{b} \Rightarrow b > a$ (as $a < 0, b < 0$) $\Rightarrow T_1 \Rightarrow T$

and $T \Rightarrow T_1$ (as $a < 0, b < 0$) $\Rightarrow b > a \Rightarrow \frac{1}{a} > \frac{1}{b}$ (as $a < 0, b < 0$)

So, for case C_2, $S \Leftrightarrow T$.

Case $\boldsymbol{C_3}$

$S: \frac{1}{a} > \frac{1}{b}$ is always true (as $a > 0, b < 0$)

and T_3 is always true

So, for case C_3, both S & T will be true.

Case $\boldsymbol{C_4}$

$S: \frac{1}{a} > \frac{1}{b}$ is never true (as $a < 0, b > 0$)

and none of the T_i are true

So, for case C_4, both S & T will be false.

So we have shown, in each case, that S & T follow suit, and therefore $S \Leftrightarrow T$.