Proof

Given that a, b & c are positive numbers, prove that $\frac{a}{b} < \frac{a+c}{b+c}$ when $a < b$.

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(1) What is wrong with the following:

$\frac{a}{b} < \frac{a+c}{b+c} \Rightarrow a(b+c) < b(a+c)$ (as $b > 0$ & $b + c > 0$)

$\Rightarrow ac < bc \Rightarrow a < b$ (as c>0)

[Given that a, b & c are positive numbers, prove that $\frac{a}{b} < \frac{a+c}{b+c}$ when $a < b$.]

(2) What is wrong with the following:

$\frac{a}{b} < \frac{a+c}{b+c} \Leftrightarrow a(b+c) < b(a+c)$ (as $b > 0$ & $b + c > 0$)

$\Leftrightarrow ac < bc \Leftrightarrow a < b$ (as c>0)

Given that a, b & c are positive numbers, prove that $\frac{a}{b} < \frac{a+c}{b+c}$ when $a < b$.]

Improved solution:

$\frac{a}{b} < \frac{a+c}{b+c} \Leftrightarrow a(b+c) < b(a+c)$ (as $b > 0$ & $b + c > 0$)

$\Leftrightarrow ac < bc \Leftrightarrow a < b$ (as c>0)

So $a < b \Rightarrow \frac{a}{b} < \frac{a+c}{b+c}$; ie $\frac{a}{b} < \frac{a+c}{b+c}$ when $a < b$, as required.

The following are equivalent:

[Given that a, b & c are positive numbers]

(a) $\frac{a}{b} < \frac{a+c}{b+c} \Leftrightarrow a < b$ [implies and is implied by]

(b) $\frac{a}{b} < \frac{a+c}{b+c}$ if and only if $a < b$ [ought to be "only if and if"]

(c) $\frac{a}{b} < \frac{a+c}{b+c}$ is a necessary & sufficient condition for $a < b$

[ought to be "is a sufficient & necessary condition for"]

Given that n is a positive integer, prove that n is odd if and only if n^2 is odd.

[Given that n is a positive integer, prove that n is odd if and only if n^2 is odd.]

Solution

Part 1: To prove that n is odd $\Rightarrow n^2$ is odd

If n is odd, then it can be written as $2m + 1$, for some integer m.

Then $n^2 = (2m + 1)^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1$,

so that n^2 is odd.

Thus n is odd $\Rightarrow n^2$ is odd; ie n is odd only if n^2 is odd.

Part 2: To prove that n^2 is odd $\Rightarrow n$ is odd

[Proof by contradiction]

If n^2 is odd, suppose that n is even. Then $n = 2m$, for some

integer m.

But then $n^2 = (2m)^2 = 4m^2$, which is divisible by 2, and so even.

This contradicts the fact that n^2 is odd, and so n must be odd.

Thus n^2 is odd $\Rightarrow n$ is odd; ie n is odd if n^2 is odd.

[Alternatively, prove that "n not odd $\Rightarrow n^2$ is not odd"]

So n is odd if and only if n^2 is odd.

If $x > 1$, show that $x - \sqrt{x^2 - 1} < 1$

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Solution

Suppose that $x - \sqrt{x^2 - 1} \geq 1$ (*)

Then $x - 1 \geq \sqrt{x^2 - 1}$

and so, as the RHS is non-negative, $(x - 1)^2 \geq x^2 - 1$

$\Rightarrow -2x + 1 \geq -1$

$\Rightarrow 2 \geq 2x$

$\Rightarrow x \leq 1$, which contradicts the fact that $x > 1$.

Thus (*) is not possible, and so $x - \sqrt{x^2 - 1} < 1$.

Show that if $X > 1$ & $Y > 1$, then $X + Y < XY + 1$

[Show that if $X > 1$ & $Y > 1$, then $X + Y < XY + 1$]

Solution

$X + Y < XY + 1 \Leftrightarrow X + Y - XY - 1 < 0$

$\Leftrightarrow X(1 - Y) + Y - 1 < 0$

$\Leftrightarrow (X - 1)(1 - Y) < 0$

$\Leftrightarrow (X - 1)(Y - 1) > 0$

Then $X > 1$ & $Y > 1 \Rightarrow (X - 1)(Y - 1) > 0 \Rightarrow X + Y < XY + 1$

Devices for "If and only if proofs"

eg "A & B are true if and only if C, D & E are true" (*)

Device 1: Suppose that $A \Leftrightarrow C$ can be proved.

Then (*) reduces to "B is true if and only if D & E are true"

Device 2: It may be possible to assert that A and C are equivalent.

(eg "a quadratic equation has repeated roots" ≡ "$b^2 = 4ac$")

Device 3: Show that $A \Rightarrow C$ and $C \Rightarrow A$

Device 4: Show that "$B \Rightarrow D$ & E are true" and that (eg) "B not true ⇒ at least one of D & E is not true"

Device 5: Break down B not true into different cases

(eg for distinct roots: (i) one root equal to 0 (ii) both negative (iii) both positive) (iv) one negative & one positive)